PHILOSOPHICAL
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OF THE
ROYAL SOCIETY
OF
LONDON.

FOR THE YEAR MDCCCXXXII.

PART I.

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ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions*, take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion,

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as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped, that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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APPENDIX.

Meteorological Journal kept at the Apartments of the Royal Society, by order of the President and Council.

The President and Council of the Royal Society adjudged the Copley Medal for the year 1831—

To George Biddell Airy, Esq. M.A. Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge, for his papers 'On the Principle of the Construction of the Achromatic Eye-pieces of Telescopes,'—'On the Spherical Aberration of the Eye-pieces of Telescopes,'—and for other papers on Optical Subjects in the Transactions of the Cambridge Philosophical Society.

PHILOSOPHICAL TRANSACTIONS.

I.—*Researches in Physical Astronomy.* *By* J. W. Lubbock, *Esq. V.P. and Treas. R.S.*

Read November 17, 1831.

On the Theory of the Moon.

In the following paper I have given the developments which are required in the Theory of the Moon when the square of the disturbing function is retained. These expressions result from the multiplication of series, each consisting of many terms; but they are formed with great facility by means of the second Table given in my former paper on the Lunar Theory.

I have not attempted the numerical calculation of the coefficients of the inequalities according to the method here explained, at least in the second approximation; but this work, which would tend to perfect the Tables of the Moon, is a desideratum in physical astronomy. The calculations will not I think be found longer than in the method of Clairaut, nor than those which are required in several astronomical problems. The developments which I have given ought however to be verified in the first instance, although I have taken great pains to ensure their accuracy.

With respect to the convergence of the expressions, it may be remarked that when the same powers of the eccentricities are retained, the results must be identical, whichever method be employed. If part of the coefficients of the terms already considered due to the higher powers of the eccentricities are sensible, it follows that other arguments must be considered in addition to those introduced by M. Damoiseau; and conversely if the arguments which

M. **Damoiseau** has considered are sufficient, it is unnecessary in either method to carry the approximation beyond the fourth power of the eccentricity of the Moon, and quantities of that order.

The method I have employed is equally advantageous in the first approximation. I have given in conclusion the numerical results which are obtained of the coefficients of the principal inequalities when the square of the disturbing function is not considered, which may be regarded as an elementary Theory of the Moon; for the differential equations and the equations which serve to determine the coefficients retain nearly the same form in the further approximations.

The coefficient of the *variation* obtained in this manner differs only by a few seconds from that given by **Newton** in the third volume of the Principia; that of the *evection* agrees closely with the value assigned to it by M. **Damoiseau**. This latter agreement of course can only be looked upon as accidental.

Developments required for the integration of the equation

$$\frac{d^2 r^2}{2\,d t^2} - \frac{d^2 r^3 \delta \frac{1}{r}}{d t^2} + \frac{3\,d^2 . r^4 \left(\delta \frac{1}{r}\right)^2}{2\,d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d R + r\left(\frac{d R}{d r}\right) = 0$$

when the square of the disturbing force is retained.

Since $r = \underset{[0]}{1 + \frac{e^2}{2}} - \underset{[2]}{e\left(1 - \frac{3}{8}e^2\right)\cos x} - \underset{[8]}{\frac{e^2}{2}\left(1 - \frac{2e^2}{3}\right)\cos 2x} - \underset{[20]}{\frac{3e^3}{8}\cos 3x} - \underset{[38]}{\frac{e^4}{3}\cos 4x}$

$$r\,\delta\frac{1}{r} = \left\{\left(1 + \frac{e^2}{2}\right)r_1 - \frac{e^2}{2}\left(1 - \frac{3}{8}e^2\right)\left\{r_3 + r_4\right\} - \frac{e^4}{4}\left\{r_9 + r_{10}\right\}\right\}\underset{[1]}{\cos 2t}$$

$$+\left\{\left(1 + \frac{e^2}{2}\right)r_2 - \frac{1}{2}\left(1 - \frac{3}{8}e^2\right)\left\{2r_0 + e^2 r_8\right\} - \frac{e^2}{4}r_2\right\}\underset{[2]}{e\cos x}$$

$$+\left\{\left(1 + \frac{e^2}{2}\right)r_3 - \frac{1}{2}\left(1 - \frac{3}{8}e^2\right)\left\{e^2 r_9 + r_1\right\} - \frac{e^2}{4}r_4\right\}\underset{[3]}{e\cos(2t - x)}$$

$$+\left\{\left(1 + \frac{e^2}{2}\right)r_4 - \frac{1}{2}\left(1 - \frac{3}{8}e^2\right)\left\{r_1 + e^2 r_{10}\right\} - \frac{e^2}{4}r_3\right\}\underset{[4]}{e\cos(2t + x)}$$

$$+\left\{\left(1 + \frac{e^2}{2}\right)r_5 - \frac{1}{2}\left(1 - \frac{3}{8}e^2\right)\left\{e^2 r_{14} + e^2 r_{11}\right\}\right\}\underset{[5]}{e_{\iota}\cos z}$$

$$+\left\{\left(1+\frac{e^2}{2}\right)r_6-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{12}+e^2r_{16}\right\}\right\}e_{\prime}\cos(2t-z)$$

[6]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_7-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{15}+e^2r_{13}\right\}\right\}e_{\prime}\cos(2t+z)$$

[7]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_8-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{r_2+e^2r_{20}\right\}-\frac{3}{16}e^2r_2\right\}e^2\cos 2x$$

[8]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_9-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{21}+r_3\right\}-\frac{r_1}{4}-\frac{3}{16}e^2r_4\right\}e^2\cos(2t-2x)$$

[9]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{10}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{r_4+e^2r_{22}\right\}-\frac{r_1}{4}-\frac{3}{16}e^2r_3\right\}e^2\cos(2t+2x)$$

[10]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{11}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{r_5+e^2r_{23}\right\}-\frac{e^2}{4}r_{14}\right\}e\,e_{\prime}\cos(x+z)$$

[11]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{12}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{24}+r_6\right\}-\frac{e^2}{4}r_{16}\right\}e\,e_{\prime}\cos(2t-x-z)$$

[12]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{13}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{r_7+e^2r_{25}\right\}-\frac{e^2}{4}r_{15}\right\}e\,e_{\prime}\cos(2t+x+z)$$

[13]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{14}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{26}+r_5\right\}\right\}e\,e_{\prime}\cos(x-z)$$

[14]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{15}-\frac{1}{2}\left(1-\frac{3}{8}e^2\right)\left\{e^2r_{27}+r_7\right\}\right\}e\,e_{\prime}\cos(2t-x+z)$$

[15]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{16}-\frac{1}{2}\left\{1-\frac{3}{8}e^2\right\}\left\{r_6+e^2r_{28}\right\}\right\}e\,e_{\prime}\cos(2t+x-z)$$

[16]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{17}-\frac{1}{2}\left\{1-\frac{3}{8}e^2\right\}\left\{e^2r_{32}+e^2r_{29}\right\}\right\}e_{\prime}^2\cos 2z$$

[17]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{18}-\frac{1}{2}\left\{1-\frac{3}{8}e^2\right\}\left\{e^2r_{30}+e^2r_{34}\right\}\right\}e_{\prime}^2\cos(2t-2z)$$

[18]

$$+\left\{\left(1+\frac{e^2}{2}\right)r_{19}-\frac{1}{2}\left\{1-\frac{3}{8}e^2\right\}\left\{e^2r_{33}+e^2r_{31}\right\}\right\}e_{\prime}^2\cos(2t+2z)$$

[19]

+ &c. &c.

From the preceding development, that of $r^3\,\delta\,.\,\frac{1}{r}$ may be immediately inferred.

$$\mathrm{r}^3 = \underset{[0]}{1 + 3\,e^2\left(1 + \frac{e^2}{8}\right)} - \underset{[2]}{3\,e\left(1 + \frac{3}{8}\,e^2\right)\cos x} - \underset{[8]}{\frac{5}{8}\,e^4\cos 2\,x} + \underset{[20]}{\frac{e^3}{8}\cos 3\,x} + \underset{[38]}{\frac{e^4}{8}\cos 4\,x}$$

The following approximate value of $\mathrm{r}\,\delta\,\frac{1}{r}$ will probably be found sufficient.

$$\mathrm{r}\,\delta\,\frac{1}{r} = \underset{[1]}{\left\{\left(1 + \frac{e^2}{2}\right)r_1 - \frac{e^2}{2}\left(r_3 + r_4\right)\right\}\cos 2\,t} + \underset{[2]}{\left\{r_2 - r_0\right\}e\cos x}$$

$$+ \underset{[3]}{\left\{r_3 - \frac{r_1}{2}\right\}e\cos(2\,t - x)} + \underset{[4]}{\left\{r_4 - \frac{r_1}{2}\right\}e\cos(2\,t + x)}$$

$$+ \underset{[5]}{r_5\,e_{,}\cos z} + \underset{[6]}{r_6\,e_{,}\cos(2\,t - z)} + \underset{[7]}{r_7\,e_{,}\cos(2\,t + z)} + \underset{[8]}{\left\{r_8 - \frac{r_1}{2}\right\}e^2\cos 2\,x}$$

$$+ \underset{[9]}{\left\{r_9 - \frac{r_3}{2} - \frac{r_1}{4}\right\}e^2\cos(2\,t - 2\,x)} + \underset{[10]}{\left\{r_{10} - \frac{r_4}{2} - \frac{r_1}{4}\right\}e^2\cos(2\,t + 2\,x)}$$

$$+ \underset{[11]}{\left\{r_{11} - \frac{r_5}{2}\right\}e\,e_{,}\cos(x + z)} + \underset{[12]}{\left\{r_{12} - \frac{r_6}{2}\right\}e\,e_{,}\cos(2\,t - x - z)}$$

$$+ \underset{[13]}{\left\{r_{13} - \frac{r_7}{2}\right\}e\,e_{,}\cos(2\,t + x + z)} + \underset{[14]}{\left\{r_{14} - \frac{r_5}{2}\right\}e\,e_{,}\cos(x - z)}$$

$$+ \underset{[15]}{\left\{r_{15} - \frac{r_7}{2}\right\}e\,e_{,}\cos(2\,t - x + z)} + \underset{[16]}{\left\{r_{16} - \frac{r_6}{2}\right\}e\,e_{,}\cos(2\,t + x - z)}$$

$$+ \underset{[17]}{r_{17}\,e_{,}^2\cos 2\,z} + \underset{[18]}{r_{18}\,e_{,}^2\cos(2\,t - 2\,z)} + \underset{[19]}{r_{19}\,e_{,}^2\cos(2\,t + 2\,z)}$$

$$a^2\left(\delta\,\frac{1}{r}\right)^2 = \underset{[0]}{r_0^2 + \frac{r_1^2}{2} + \frac{e^2\,r_2^2}{2} + \frac{e^2\,r_3^2}{2} + \frac{e^2\,r_4^2}{2} + \frac{e_{,}^2\,r_5^2}{2} + \frac{e_{,}^2\,r_6^2}{2} + \frac{e_{,}^2\,r_7^2}{2}}$$

$$+ \underset{[1]}{\{2\,r_0\,r_1 + e^2\,(r_3 + r_4)\,r_2 + e_{,}^2\,(r_6 + r_7)\,r_5\}\cos 2\,t} + \underset{[2]}{\{(r_4 + r_3)\,r_1 + 2\,r_0\,r_2\}\,e\cos x}$$

$$+ \underset{[3]}{\{r_1\,r_2 + 2\,r_0\,r_3\}\,e\cos(2\,t - x)} + \underset{[4]}{\{r_1\,r_2 + 2\,r_0\,r_4\}\cos(2\,t + x)}$$

$$+ \underset{[5]}{\{r_1\,r_7 + r_1\,r_6 + 2\,r_0\,r_5\}\,e_{,}\cos z} + \underset{[6]}{\{r_5\,r_1 + 2\,r_0\,r_6\}\,e_{,}\cos(2\,t - z)}$$

$$+ \{r_5 r_1 + 2 r_0 r_7\} \underset{[7]}{e_{,} \cos (2t + z)} + \{r_2^2 + r_4 r_3 + r_1 r_9 + r_1 r_{10}\} \underset{[8]}{e^2 \cos 2x}$$

$$+ \{r_2 r_3 + 2 r_0 r_9\} \underset{[9]}{e^2 \cos (2t - 2x)} + \{r_4 r_8 + 2 r_0 r_{10}\} \underset{[10]}{e^2 \cos (2t + 2x)}$$

$$+ \{r_1 r_{13} + r_1 r_{12} + r_2 r_5 + r_6 r_4 + r_3 r_7 + 2 r_0 r_{11}\} \underset{[11]}{e e_{,} \cos (x + z)}$$

$$+ \{r_{11} r_1 + r_2 r_6 + r_5 r_3 + 2 r_0 r_{12}\} \underset{[12]}{e e_{,} \cos (2t - x - z)}$$

$$+ \{r_{11} r_1 + r_2 r_7 + r_5 r_4 + 2 r_0 r_{13}\} \underset{[13]}{e e_{,} \cos (2t + x + z)}$$

$$+ \{r_{16} r_1 + r_{15} r_1 + r_2 r_5 + r_6 r_3 + r_7 r_4 + 2 r_0 r_{14}\} \underset{[14]}{e e_{,} \cos (x - z)}$$

$$+ \{r_{14} r_1 + r_2 r_7 + r_5 r_3\} \underset{[15]}{e e_{,} \cos (2t - x + z)} + \{r_{14} r_1 + r_2 r_6 + r_5 r_4\} \underset{[16]}{e e_{,} \cos (2t + x - z)}$$

$$+ \{r_5^2 + r_7 r_6 + r_1 r_{18} + r_1 r_{19}\} \underset{[17]}{e_{,}^2 \cos 2z} + \{r_{17} r_1 + r_5 r_6\} \underset{[18]}{e_{,}^2 \cos (2t - 2z)}$$

$$+ \{r_{17} r_1 + r_7 r_5\} \underset{[19]}{e_{,}^2 \cos (2t + 2z)} + \underset{[131]}{\frac{r_1^2}{2} \cos 4t} + \underset{[132]}{\frac{r_3^2}{2} \cos (4t - 2x)}$$

From the preceding development that of $r^4 \left(\delta \frac{1}{r}\right)^2$ may be easily inferred.

$$r^4 = a^4 \{\underset{[0]}{1 + 5e^2} - \underset{[2]}{4 e \cos x} + \underset{[8]}{e^2 \cos 2x}\}$$

Considering the terms only in R multiplied by $\frac{a^2}{a_{,}^3}$

$$R = - m_{,} \left\{ \frac{r'^2}{4 r_{,}^3} \{1 + 3 \cos (2\lambda - 2\lambda_{,}) - 2 s^2\} \right\}$$

$$= - m_{,} \left\{ \frac{r^2}{4 (1 + s^2) r_{,}^3} \{1 + 3 \cos (2\lambda - 2\lambda_{,}) - 2 s^2\} \right\}$$

neglecting s^4

$$= - m_{,} \left\{ \frac{r^2}{4 r_{,}^3} \{1 + 3 \cos (2\lambda - 2\lambda_{,})\} \{1 - s^2\} - \frac{r^2}{2 r_{,}^3} s^2 \right\}$$

$$\frac{dR}{ds} = m_{,} \left\{ \frac{r^2}{r_{,}^3} + \frac{r^2}{2 r_{,}^3} \{1 + 3 \cos (2\lambda - 2\lambda_{,})\} \right\} s.$$

$$\frac{r^2}{2\,r_{,}^3}{}^{*} + \frac{r^2}{4\,r_{,}^3}\{1 + 3\cos(2\lambda - 2\lambda_{,})\}$$

$$= \frac{a^2}{a_{,}^3}\left\{\frac{3}{4}\left\{1 + \frac{3}{2}e^2 + \frac{3}{2}e_{,}^2\right\} + \frac{3}{4}\left\{1 - \frac{5}{2}e^2 - \frac{5}{2}e_{,}^2\right\}\cos 2t - \frac{3}{2}e\cos x\right.$$

$$- \frac{9}{4}e\cos(2t - x) + \frac{3}{4}e\cos(2t + x) + \frac{9}{4}e_{,}\cos z + \frac{21}{8}e_{,}\cos(2t - z)$$

$$- \frac{3}{8}e_{,}\cos(2t + z) - \frac{3}{8}e^2\cos 2x + \frac{15}{8}e^2\cos(2t - 2x)$$

$$+ \frac{3}{4}e^2\cos(2t + 2x) - \frac{9}{4}ee_{,}\cos(x + z) - \frac{63}{8}ee_{,}\cos(2t - x - z)$$

$$- \frac{3}{8}ee_{,}\cos(2t + x + z) - \frac{9}{4}ee_{,}\cos(x - z) + \frac{9}{8}ee_{,}\cos(2t - x + z)$$

$$+ \frac{21}{8}ee_{,}\cos(2t + x - z) + \frac{27}{8}e_{,}^2\cos 2z + \frac{51}{8}e_{,}^2\cos(2t - 2z)$$

$$\frac{dR}{ds} = \frac{m_{,}a^2}{a_{,}^3}\left\{\underset{[146]}{\frac{204}{137}\gamma\sin y} - \underset{[147]}{\frac{20}{27}\gamma\sin(2t - y)} + \underset{[148]}{\frac{20}{27}\sin(2t + y)} + \underset{[149]}{\frac{3}{2}e\sin(x - y)}\right.$$

$$- \underset{[150]}{\frac{3}{2}e(\sin x + y)} + \underset{[151]}{\frac{9}{4}e\gamma\sin(2t - x - y)} - \underset{[152]}{\frac{9}{4}e\gamma\sin(2t - x + y)}$$

$$- \underset{[153]}{\frac{3}{4}e\gamma\sin(2t + x - y)} + \underset{[154]}{\frac{3}{4}e\gamma\sin(2t + x + y)} - \underset{[155]}{\frac{9}{4}e_{,}\gamma\sin(z - y)}$$

$$+ \underset{[156]}{\frac{9}{4}e_{,}\gamma\sin(z + y)} - \underset{[157]}{\frac{21}{8}e_{,}\gamma\sin(2t - z - y)} + \underset{[158]}{\frac{21}{8}e_{,}\gamma\sin(2t - z + y)}$$

$$+ \underset{[159]}{\frac{3}{8}e_{,}\gamma\sin(2t + z - y)} - \underset{[160]}{\frac{3}{8}e_{,}\sin(2t + z + y)} + \underset{[161]}{\frac{3}{8}e^2\gamma\sin(2x - y)}$$

$$- \underset{[162]}{\frac{3}{8}e^2\gamma\sin(2x + y)} - \underset{[163]}{\frac{15}{8}e^2\gamma\sin(2t - 2x - y)} + \underset{[164]}{\frac{15}{8}e^2\gamma\sin(2t - 2x + y)}$$

$$- \underset{[165]}{\frac{3}{4}e^2\gamma\sin(2t + 2x - y)} + \underset{(166)}{\frac{3}{4}e^2\gamma\sin(2t + 2x + y)}$$

$$+ \underset{[167]}{\frac{9}{4}ee_{,}\gamma\sin(x + z - y)} - \underset{[168]}{\frac{9}{4}ee_{,}\gamma\sin(x + z + y)}$$

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* See Phil. Trans. 1831, p. 255 and 263.

$$+\underset{[169]}{\frac{63}{8}ee_{,}\gamma\sin(2t-x-z-y)}-\underset{[170]}{\frac{63}{8}ee_{,}\gamma\sin(2t-x-z+y)}$$

$$+\underset{[171]}{\frac{3}{8}ee_{,}\gamma\sin(2t+x+z-y)}-\underset{[172]}{\frac{3}{8}ee_{,}\gamma\sin(2t+x+z+y)}$$

$$+\underset{[173]}{\frac{9}{4}ee_{,}\gamma\sin(x-z-y)}-\underset{[174]}{\frac{9}{4}ee_{,}\gamma\sin(x-z+y)}$$

$$-\underset{[175]}{\frac{9}{8}ee_{,}\gamma\sin(2t-x+z-y)}+\underset{[176]}{\frac{9}{8}ee_{,}\gamma\sin(2t-x+z+y)}$$

$$-\underset{[177]}{\frac{21}{8}ee_{,}\gamma\sin(2t+x-z-y)}+\underset{[178]}{\frac{21}{8}ee_{,}\gamma\sin(2t+x-z+y)}$$

$$-\underset{[179]}{\frac{27}{8}e_{,}^{2}\gamma\sin(2z-y)}+\underset{[180]}{\frac{27}{8}e_{,}^{2}\gamma\sin(2z+y)}-\underset{[181]}{\frac{51}{8}e_{,}^{2}\gamma\sin(2t-2z-y)}$$

$$+\underset{[182]}{\frac{51}{8}e_{,}^{2}\gamma\sin(2t-2z+y)}$$

The inequality of latitude of which the argument is $2t-y$ being far greater than the rest, $\delta s=\gamma s_{147}\sin(2t-y)$ nearly.

If $\quad e=\cdot 0548442 \qquad e_{,}=\cdot 0167927 \qquad \gamma=\cdot 0900684$

See Mém. sur la Théorie de la Lune, p. 502.

$$R=\frac{m_{,}a^{2}}{a_{,}^{3}}\Big\{-\underset{[0]}{9{\cdot}3947865}-\underset{[1]}{9{\cdot}8697237\cos 2t}+\underset{[2]}{9{\cdot}6933013\,e\cos x}$$

$$+\underset{[3]}{0{\cdot}3494165\,e\cos(2t-x)}-\underset{[4]}{9{\cdot}8698883\,e\cos(2t+x)}$$

$$-\underset{[5]}{9{\cdot}8718614\,e_{,}\cos z}-\underset{[6]}{9{\cdot}4138294\,e_{,}\cos(2t-z)}$$

$$+\underset{[7]}{9{\cdot}5685221\,e_{,}\cos(2t+z)}+\underset{[8]}{9{\cdot}0917777\,e^{2}\cos 2x}$$

$$-\underset{[9]}{0{\cdot}2709438\,e^{2}\cos(2t-2x)}-\underset{[10]}{9{\cdot}8697180\,e^{2}\cos(2t+2x)}$$

$$+\underset{[11]}{9{\cdot}8697237\,ee_{,}\cos(x+z)}+\underset{[12]}{0{\cdot}8935219\,ee_{,}\cos(2t-x-z)}$$

$$+ \underset{[13]}{9{\cdot}5691515\, e e_{,} \cos(2t + x + z)} + \underset{[14]}{9{\cdot}8697180\, e e_{,} \cos(x - z)}$$

$$- \underset{[15]}{0{\cdot}0486780\, e e_{,} \cos(2t - x + z)} + \underset{[16]}{0{\cdot}4139940\, e e_{,} \cos(2t + x - z)}$$

$$- \underset{[17]}{0{\cdot}0479097\, e_{,}^{2} \cos 2z} - \underset{[18]}{0{\cdot}7991728\, e_{,}^{2} \cos(2t - 2z)}$$

$$- \underset{[62]}{9{\cdot}5709386\, \gamma^{2} \cos 2z} - \underset{[63]}{9{\cdot}5761195\, \gamma^{2} \cos(2t - 2y)}$$

where the logarithms of the coefficients are written instead of the coefficients themselves.

$$R = \frac{m_{,} a^{2}}{a_{,}^{3}} \Big\{ - \underset{[0]}{\frac{34}{137}} - \underset{[1]}{\frac{20}{27} \cos 2t} + \underset{[2]}{\frac{38}{77} e \cos x} + \underset{[3]}{\frac{38}{17} e \cos(2t - x)} - \underset{[4]}{\frac{20}{27} e \cos(2t + z)}$$

$$- \underset{[5]}{\frac{32}{43} e_{,} \cos z} - \underset{[6]}{\frac{70}{27} e_{,} \cos(2t - z)} + \underset{[7]}{\frac{10}{27} e_{,} \cos(2t + z)} + \underset{[8]}{\frac{10}{81} e^{2} \cos 2x}$$

$$- \underset{[9]}{\frac{28}{15} e^{2} \cos(2t - 2x)} - \underset{[10]}{\frac{20}{27} e^{2} \cos(2t + 2x)} + \underset{[11]}{\frac{20}{27} e e_{,} \cos(x + z)}$$

$$+ \underset{[12]}{\frac{180}{23} e e_{,} \cos(2t - x - z)} + \underset{[13]}{\frac{10}{27} e e_{,} \cos(2t + x + z)} + \underset{[14]}{\frac{20}{27} e e_{,} \cos(x + z)}$$

$$- \underset{[15]}{\frac{66}{59} e e_{,} \cos(2t - x + z)} - \underset{[16]}{\frac{83}{32} e e_{,} \cos(2t + x - z)} - \underset{[17]}{\frac{67}{60} e_{,}^{2} \cos 2z}$$

$$- \underset{[18]}{\frac{233}{37} e_{,}^{2} \cos(2t - 2z)} - \underset{[62]}{\frac{16}{43} \gamma^{2} \cos 2y} - \underset{[63]}{\frac{26}{69} \gamma^{2} \cos(2t - 2y)} \text{ nearly.}$$

I make use of these approximate coefficients in the following development solely in order that it may occupy less space.

$$\delta R^{*} = \frac{m_{,} a^{2}}{a_{,}^{3}} \left\{ \frac{68}{137} r_{0}'^{\dagger} + \frac{20}{27} \left\{ r_{1}' + \lambda_{1} \right\} - \frac{38}{77} e^{2} r_{2}' - \frac{38}{17} e^{2} \left\{ r_{3}' + \lambda_{3} \right\} + \frac{20}{27} e^{2} \left\{ r_{4}' + \lambda_{4} \right\} \right.$$

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* See Phil. Trans. 1831, p. 275.

† $r \delta \frac{1}{r} = \underset{[0]}{r_{0}'} + \underset{[1]}{r_{1}' \cos 2t} + \underset{[2]}{e r_{2}' \cos x} + \underset{[3]}{e r_{3}' \cos(2t - x)}$ &c.

$\delta \lambda = \underset{[1]}{\lambda_{1} \sin 2t} + \underset{[3]}{e \lambda_{3} \sin(2t - x)}$ + &c.

Development of δR.

$$+\frac{32}{43}e_{,}^{2}r_{5}'+\frac{70}{27}e_{,}^{2}\left\{r_{6}'+\lambda_{6}\right\}-\frac{10}{27}e_{,}^{2}\left\{r_{7}'+\lambda_{7}\right\}-\frac{10}{27}\gamma^{2}s_{147}$$

[0]

$$\begin{aligned}+\Big\{&+\frac{40}{27}r_{0}'+\frac{68}{137}r_{1}'-\frac{38}{17}e^{2}r_{2}'+\frac{20}{27}e^{2}r_{2}'-\frac{38}{77}e^{2}r_{3}'-\frac{38}{77}e^{2}r_{4}'+\frac{70}{27}e_{,}^{2}\left\{r_{5}'-\lambda_{5}\right\}\\&-\frac{10}{27}e_{,}^{2}\left\{r_{5}'+\lambda_{5}\right\}+\frac{32}{43}e_{,}^{2}r_{6}'+\frac{32}{43}e_{,}^{2}r_{7}'+\frac{28}{15}e^{4}\left\{r_{8}'-\lambda_{8}\right\}+\frac{20}{27}e^{4}\left\{r_{8}'+\lambda_{8}\right\}\\&-\frac{10}{81}e^{4}r_{9}'-\frac{10}{81}e^{4}r_{10}'-\frac{180}{23}e^{2}e_{,}^{2}\left\{r_{11}'-\lambda_{11}\right\}-\frac{10}{27}e^{2}e_{,}^{2}\left\{r_{11}'+\lambda_{11}\right\}\\&-\frac{20}{27}e^{2}e_{,}^{2}r_{12}'+\frac{66}{59}e^{2}e_{,}^{2}\left\{r_{14}'-\lambda_{14}\right\}+\frac{83}{32}e^{2}e_{,}^{2}\left\{r_{14}'+\lambda_{14}\right\}\\&-\frac{102}{137}\gamma^{2}s_{147}\Big\}\cos 2t\end{aligned}$$

[1]

$$\begin{aligned}+\Big\{&-\frac{76}{77}r_{0}'+\frac{20}{27}\left\{r_{1}'+\lambda_{1}\right\}-\frac{38}{17}\left\{r_{1}'+\lambda_{1}\right\}+\frac{68}{137}r_{2}'+\frac{28}{15}e^{2}\left\{r_{3}'+\lambda_{3}\right\}\\&+\frac{20}{27}\left\{r_{3}'+\lambda_{3}\right\}+\frac{20}{27}\left\{r_{4}'+\lambda_{4}\right\}-\frac{20}{27}e_{,}^{2}r_{5}'-\frac{20}{27}e_{,}^{2}r_{5}'-\frac{3}{8}\gamma^{2}s_{147}\\&+\frac{9}{8}\gamma^{2}s_{147}\Big\}e\cos x\end{aligned}$$

[2]

$$\begin{aligned}+\Big\{&-\frac{76}{17}r_{0}'-\frac{38}{77}r_{1}'+\frac{20}{27}r_{2}'+\frac{68}{137}r_{3}'-\frac{180}{23}e_{,}^{2}\left\{r_{5}'-\lambda_{5}\right\}+\frac{66}{59}e_{,}^{2}\left\{r_{5}'+\lambda_{5}\right\}\\&+\frac{3}{4}\gamma^{2}s_{147}\Big\}e\cos(2t-x)\end{aligned}$$

[3]

$$\begin{aligned}+\Big\{&+\frac{40}{27}r_{0}'-\frac{38}{77}r_{1}'+\frac{20}{27}r_{2}'-\frac{10}{81}e^{2}r_{3}'+\frac{68}{137}r_{4}'+\frac{83}{32}e_{,}^{2}\left\{r_{5}'-\lambda_{5}\right\}-\frac{10}{27}e_{,}^{2}\left\{r_{5}'+\lambda_{5}\right\}\\&+\frac{3}{4}\gamma^{2}s_{147}\Big\}e\cos(2t+x)\end{aligned}$$

[4]

$$\begin{aligned}+\Big\{&+\frac{64}{43}r_{0}'-\frac{10}{27}\left\{r_{1}'+\lambda_{1}\right\}+\frac{70}{27}\left\{r_{1}'+\lambda_{1}\right\}+\frac{66}{59}e^{2}\left\{r_{3}'+\lambda_{3}\right\}-\frac{180}{23}e^{2}\left\{r_{3}'+\lambda_{3}\right\}\\&+\frac{68}{137}r_{5}'+\frac{67}{60}e_{,}^{2}r_{5}'-\frac{3}{16}\gamma^{2}s_{147}-\frac{21}{16}\gamma^{2}s_{147}\Big\}e_{,}\cos z\end{aligned}$$

[5]

$$\begin{aligned}+\Big\{&+\frac{140}{27}r_{0}'+\frac{32}{43}r_{1}'-\frac{20}{27}e^{2}r_{3}'+\frac{233}{37}e_{,}^{2}\left\{r_{5}'-\lambda_{5}\right\}+\frac{20}{27}\left\{r_{5}'+\lambda_{5}\right\}+\frac{68}{137}r_{6}'\\&-\frac{9}{8}\gamma^{2}s_{147}\Big\}e_{,}\cos(2t-z)\end{aligned}$$

[6]

Development of δR.

$$+\left\{-\frac{20}{27}r_0'+\frac{32}{43}r_1'-\frac{20}{27}e^2r_3'+\frac{20}{27}\left\{r_5'-\lambda_5\right\}+\frac{68}{137}r_7'-\frac{9}{8}\gamma^2s_{147}\right\}e_{,}\cos(2t+z)$$

[7]

$$\begin{aligned}+\Big\{&-\frac{20}{81}r_0'+\frac{20}{27}\left\{r_1'+\lambda_1\right\}+\frac{28}{15}\left\{r_1'+\lambda_1\right\}-\frac{7}{32}e^2\left\{r_3'+\lambda_3\right\}+\frac{20}{27}\left\{r_3'+\lambda_3\right\}\\&-\frac{38}{17}\left\{r_4'+\lambda_4\right\}-\frac{3}{16}e_{,}^2r_5'-\frac{3}{16}e_{,}^2r_3'+\frac{68}{137}r_8'+\frac{20}{27}\left\{r_9'+\lambda_9\right\}\\&+\frac{20}{27}\left\{r_{10}'+\lambda_{10}\right\}-\frac{3}{8}\gamma^2s_{147}-\frac{15}{16}\gamma^2s_{147}\Big\}e^2\cos 2x\end{aligned}$$

[8]

$$\begin{aligned}+\Big\{&+\frac{56}{15}r_0'-\frac{10}{81}r_1'-\frac{38}{17}r_2'-\frac{38}{77}r_3'+\frac{105}{16}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{15}{16}e_{,}^2\left\{r_5'+\lambda_5\right\}\\&+\frac{20}{27}\left\{r_8'+\lambda_8\right\}+\frac{68}{137}r_9'+\frac{3}{16}\gamma^2s_{147}\Big\}e^2\cos(2t-2x)\end{aligned}$$

[9]

$$\begin{aligned}+\Big\{&+\frac{40}{27}r_0'-\frac{10}{81}r_1'+\frac{20}{27}r_2'-\frac{1}{16}e^2r_3'-\frac{38}{77}r_4'+\frac{21}{8}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{3}{8}e_{,}^2\left\{r_5'+\lambda_5\right\}\\&+\frac{20}{27}\left\{r_8'-\lambda_8\right\}+\frac{68}{137}r_{10}'+\frac{3}{16}\gamma^2s_{147}\Big\}e^2\cos(2t+2x)\end{aligned}$$

[10]

$$\begin{aligned}+\Big\{&-\frac{40}{27}r_0'-\frac{10}{27}\left\{r_1'+\lambda_1\right\}-\frac{180}{23}\left\{r_1'+\lambda_1\right\}+\frac{105}{16}\left\{r_3'+\lambda_3\right\}-\frac{10}{27}\left\{r_3'+\lambda_3\right\}\\&+\frac{70}{27}\left\{r_4'+\lambda_4\right\}-\frac{38}{77}r_5'-\frac{9}{8}e_{,}^2r_5'+\frac{20}{27}\left\{r_6'+\lambda_6\right\}-\frac{38}{17}\left\{r_7'+\lambda_7\right\}\\&-\frac{20}{27}e^2r_8'+\frac{66}{59}e^2\left\{r_9'+\lambda_9\right\}+\frac{83}{32}e^2\left\{r_{10}'+\lambda_{10}\right\}+\frac{68}{137}r_{11}'+\frac{20}{27}\left\{r_{12}'+\lambda_{12}\right\}\\&-\frac{10}{81}e^2r_{14}'+\frac{3}{16}\gamma^2s_{147}+\frac{63}{16}\gamma^2s_{147}\Big\}ee_{,}\cos(x+z)\end{aligned}$$

[11]

$$\begin{aligned}+\Big\{&-\frac{360}{23}r_0'-\frac{20}{27}r_1'+\frac{70}{27}r_2'+\frac{32}{43}r_3'-\frac{153}{8}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{38}{17}\left\{r_5'+\lambda_5\right\}-\frac{38}{77}r_6'\\&+\frac{20}{27}\left\{r_{11}'+\lambda_{11}\right\}+\frac{68}{137}r_{12}'+\frac{9}{8}\gamma^2s_{147}\Big\}ee_{,}\cos(2t-x-z)\end{aligned}$$

[12]

$$\begin{aligned}+\Big\{&-\frac{20}{27}r_0'-\frac{20}{27}r_1'-\frac{10}{27}r_2'-\frac{3}{16}e^2\left\{r_3'-\lambda_3\right\}+\frac{32}{43}r_4'+\frac{20}{27}\left\{r_5'-\lambda_5\right\}-\frac{38}{77}r_7'\\&+\frac{20}{27}\left\{r_{11}'-\lambda_{11}\right\}+\frac{68}{137}r_{13}'+\frac{9}{8}\gamma^2s_{147}\Big\}ee_{,}\cos(2t+x+z)\end{aligned}$$

[13]

Development of δ *R*.

$$\begin{aligned}
&+\Bigg\{-\frac{40}{27}r_0'+\frac{83}{32}\Big\{r_1'+\lambda_1\Big\}+\frac{66}{59}\Big\{r_1'+\lambda_1\Big\}+\frac{32}{43}r_2'-\frac{15}{16}e^2\Big\{r_3'+\lambda_3\Big\}+\frac{70}{27}\Big\{r_3'+\lambda_3\Big\}\\
&\quad-\frac{10}{27}\Big\{r_4'+\lambda_4\Big\}-\frac{9}{8}e_{\prime}^2r_5'-\frac{38}{77}r_5'-\frac{38}{17}\Big\{r_6'+\lambda_6\Big\}+\frac{20}{27}\Big\{r_7'+\lambda_7\Big\}\\
&\quad-\frac{20}{27}e^2r_8-\frac{180}{23}e^2\Big\{r_9'+\lambda_9\Big\}-\frac{10}{27}e^2\Big\{r_{10}'+\lambda_{10}\Big\}-\frac{10}{81}e^2r_{11}'\\
&\quad+\frac{28}{15}e^2\Big\{r_{12}'+\lambda_{12}\Big\}+\frac{68}{137}r_{14}'+\frac{20}{27}\Big\{r_{15}'+\lambda_{15}\Big\}+\frac{20}{27}\Big\{r_{16}'+\lambda_{16}\Big\}\\
&\quad-\frac{21}{16}\gamma^2s_{147}-\frac{9}{16}\gamma^2s_{147}\Bigg\}ee_{\prime}\cos(x-z)
\end{aligned}$$

[14]

$$\begin{aligned}
&+\Bigg\{+\frac{132}{59}r_0'-\frac{20}{27}r_1'-\frac{10}{27}r_2'+\frac{32}{43}r_3'-\frac{38}{17}\Big\{r_5'-\lambda_5\Big\}+\frac{20}{27}\Big\{r_{14}'+\lambda_{14}\Big\}\\
&\quad+\frac{68}{137}r_{15}'+\frac{9}{8}\gamma^2s_{147}\Bigg\}ee_{\prime}\cos(2t-x+z)
\end{aligned}$$

[15]

$$\begin{aligned}
&+\Bigg\{+\frac{83}{16}r_0'-\frac{20}{27}r_1'+\frac{70}{27}r_2'-\frac{3}{16}e^2r_3'+\frac{32}{43}r_4'+\frac{51}{8}e_{\prime}^2\Big\{r_5'-\lambda_5\Big\}+\frac{20}{27}\Big\{r_5'+\lambda_5\Big\}\\
&\quad-\frac{38}{77}r_6'+\frac{20}{27}\Big\{r_{14}'-\lambda_{14}\Big\}+\frac{68}{137}r_{16}'+\frac{9}{8}\gamma^2s_{147}\Bigg\}ee_{\prime}\cos(2t+x-z)
\end{aligned}$$

[16]

$$\begin{aligned}
&+\Bigg\{+\frac{67}{30}r_0'+\frac{233}{37}\Big\{r_1'+\lambda_1\Big\}-\frac{153}{8}e^2\Big\{r_3'+\lambda_3\Big\}+\frac{32}{43}r_5'+\frac{53}{32}e_{\prime}^2r_5'-\frac{10}{27}\Big\{r_6'+\lambda_6\Big\}\\
&\quad+\frac{70}{27}\Big\{r_7'+\lambda_7\Big\}+\frac{68}{137}r_{17}+\frac{20}{27}\Big\{r_{18}'+\lambda_{18}\Big\}+\frac{20}{27}\Big\{r_{19}'+\lambda_{19}\Big\}\\
&\quad-\frac{51}{16}\gamma^2s_{147}\Bigg\}e_{\prime}^2\cos 2z
\end{aligned}$$

[17]

$$\begin{aligned}
&+\Bigg\{+\frac{466}{37}r_0'+\frac{67}{60}r_1'-\frac{9}{8}e^2r_3'+\frac{845}{64}e_{\prime}^2\Big\{r_5'-\lambda_5\Big\}+\frac{70}{27}\Big\{r_5'+\lambda_5\Big\}+\frac{32}{43}r_6'\\
&\quad+\frac{20}{27}\Big\{r_{17}'+\lambda_{17}\Big\}+\frac{68}{137}r_{18}'-\frac{27}{16}\gamma^2s_{147}\Bigg\}e_{\prime}^2\cos(2t-2z)
\end{aligned}$$

[18]

$$\begin{aligned}
&+\Bigg\{+\frac{67}{60}r_1'-\frac{9}{8}e^2r_3'-\frac{10}{27}\Big\{r_5'-\lambda_5\Big\}+\frac{e_{\prime}^2}{64}\Big\{r_5'+\lambda_5\Big\}+\frac{32}{43}r_7'+\frac{20}{27}\Big\{r_{17}'-\lambda_{17}\Big\}\\
&\quad+\frac{68}{137}r_{19}'-\frac{27}{16}\gamma^2s_{147}\Bigg\}e_{\prime}^2\cos(2t+2z)
\end{aligned}$$

[19]

Development of δR.

$$+\left\{+\frac{26}{69}\left\{r_1{}'+\lambda_1\right\}-\frac{3}{8}e^2\left\{r_3{}'+\lambda_3\right\}+\frac{9}{16}e_{,}^2r_5{}'+\frac{9}{16}e_{,}^2r_5{}'+\frac{10}{27}s_{147}\right\}\gamma^2\cos 2y$$
[62]

$$+\left\{+\frac{16}{43}r_1{}'-\frac{9}{8}r_3{}'+\frac{21}{16}e_{,}^2\left\{r_5{}'-\lambda_5\right\}-\frac{3}{16}e_{,}^2\left\{r_5{}'+\lambda_5\right\}+\frac{102}{137}s_{147}\right\}\gamma^2\cos(2t-2y)$$
[63]

$$+\left\{+\frac{16}{43}r_1{}'+\frac{3}{8}r_3{}'\right\}\gamma^2\cos(2t+2y)$$
[64]

$$+\left\{-\frac{3}{8}\left\{r_1{}'+\lambda_1\right\}+\frac{26}{69}\left\{r_3{}'+\lambda_3\right\}-\frac{9}{8}s_{147}\right\}\gamma^2 e\cos(x-2y)$$
[65]

$$+\left\{-\frac{3}{8}\left\{r_1{}'+\lambda_1\right\}+\frac{3}{8}s_{147}\right\}\gamma^2 e\cos(x+2y)$$
[66]

$$+\left\{+\frac{3}{8}r_1{}'+\frac{16}{43}r_3{}'-\frac{3}{4}s_{147}\right\}\gamma^2 e\cos(2t-x-2y)$$
[67]

$$+\left\{-\frac{9}{8}r_1{}'+\frac{16}{43}r'_3\right\}\gamma^2 e\cos(2t-x+2y)$$
[68]

$$+\left\{-\frac{9}{8}r_1{}'-\frac{3}{4}s_{147}\right\}\gamma^2 e\cos(2t+x-2y)+\frac{3}{8}r_1{}'\gamma^2 e\cos(2t+x+2y)$$
[69] [70]

$$+\left\{-\frac{3}{16}\{r_1{}'+\lambda_1\}+\frac{16}{43}r'_5+\frac{21}{16}s_{147}\right\}\gamma^2 e_{,}\cos(z-2y)$$
[71]

$$+\left\{+\frac{21}{16}\{r_1{}'+\lambda_1\}+\frac{16}{43}r_5{}'+\frac{3}{16}s_{147}\right\}\gamma^2 e_{,}\cos(z+2y)$$
[72]

$$+\left\{+\frac{9}{16}r_1{}'+\frac{26}{69}r_5{}'+\frac{9}{8}s_{147}\right\}\gamma^2 e_{,}\cos(2t-z-2y)$$
[73]

$$+\frac{9}{16}r_1{}'\gamma^2 e_{,}\cos(2t-z+2y)$$
[74]

$$+\left\{+\frac{9}{16}r_1{}'+\frac{26}{69}\{r_5{}'-\lambda_5\}+\frac{9}{8}s_{147}\right\}\gamma^2 e_{,}\cos(2t+z-2y)$$
[75]

$$+\frac{9}{16}r_1{}'\gamma^2 e_{,}\cos(2t+z+2y)$$
[76]

Development of δR.

$$+\left\{-\frac{3}{8}\{r_3'+\lambda_3\}+\frac{15}{16}s_{147}\right\}\gamma^2e^2\cos(2x-2y)$$

[77]

$$+\frac{3}{8}s_{147}\gamma^2e^2\cos(2x+2y)+\left\{+\frac{3}{8}r_3'-\frac{3}{16}s_{147}\right\}\gamma^2e^2\cos(2t-2x-2y)$$

[78] [79]

$$-\frac{9}{8}r_3'\gamma^2e^2\cos(2t-2x+2y)-\frac{3}{16}s_{147}\gamma^2e^2\cos(2t+2x-2y)$$

[80] [81]

$$+\left\{-\frac{3}{16}\{r_3'+\lambda_3\}-\frac{9}{8}r_5'-\frac{63}{16}s_{147}\right\}\gamma^2ee_{\prime}\cos(x+z-2y)$$

[83]

$$+\left\{+\frac{3}{8}r_5'-\frac{3}{16}s_{147}\right\}\gamma^2ee_{\prime}\cos(x+z+2y)$$

[84]

$$+\left\{+\frac{9}{16}r_3'-\frac{3}{8}\{r_5'+\lambda_5\}-\frac{9}{8}s_{147}\right\}\gamma^2ee_{\prime}\cos(2t-x-z-2y)$$

[85]

$$+\frac{9}{16}r_3'\gamma^2ee_{\prime}\cos(2t-x-z+2y)$$

[86]

$$+\left\{-\frac{3}{8}\{r_5'-\lambda_5\}-\frac{9}{8}s_{147}\right\}\gamma^2ee_{\prime}\cos(2t+x+z-2y)$$

[87]

$$+\left\{+\frac{21}{16}\{r_3'+\lambda_3\}-\frac{9}{8}r_5'+\frac{9}{16}s_{147}\right\}\gamma^2ee_{\prime}\cos(x-z-2y)$$

[89]

$$+\left\{+\frac{3}{8}r_5'+\frac{21}{16}s_{147}\right\}\gamma^2ee_{\prime}\cos(x-z+2y)$$

[90]

$$+\left\{+\frac{9}{16}r_3'-\frac{3}{8}\{r_5'-\lambda_5\}-\frac{9}{8}s_{147}\right\}\gamma^2ee_{\prime}\cos(2t-x+z-2y)$$

[91]

$$+\frac{9}{16}r_3'\gamma^2ee_{\prime}\cos(2t-x+z+2y)$$

[92]

$$+\left\{-\frac{3}{8}\{r_5'+\lambda_5\}-\frac{9}{8}s_{147}\right\}\gamma^2ee_{\prime}\cos(2t+x-z-2y)$$

[93]

$$+\left\{+\frac{9}{16}r_5'+\frac{51}{16}s_{147}\right\}\gamma^2e_{\prime}^2\cos(2z-2y)+\frac{9}{16}r_5'\gamma^2e_{\prime}^2\cos(2z-2y)$$

[95] [96]

Development of δ *R.*

$$+\frac{a}{a_{,}}\left\{+\frac{5.3}{8.2}\{r_1'+\lambda_1\}+\frac{3}{8}\left\{\frac{3}{2}r_1'+\frac{1}{2}\lambda_1\right\}-\frac{45.3}{16.2}e^2\{r_3'+\lambda_3\}\right.$$

$$\left.-\frac{15}{16}e^2\left\{\frac{3}{2}r_3'+\frac{1}{2}\lambda_3\right\}+\frac{9}{8}e_{,}^2\left\{\frac{3}{2}r_5'-\frac{1}{2}\lambda_5\right\}+\frac{3}{8}e_{,}^2\left\{\frac{3}{2}r_5'+\frac{1}{2}\lambda_5\right\}\right\}\cos t^{*}$$

[101]

$$+\frac{a}{a_{,}}\left\{-\frac{45.3}{16.2}\left\{r_1'+\lambda_1\right\}-\frac{3}{16}\left\{\frac{3}{2}r_1'+\frac{1}{2}\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{2}r_3'+\frac{1}{2}\lambda_3\right\}\right\}e\cos(t-x)$$

[102]

$$+\frac{a}{a_{,}}\left\{+\frac{15.3}{16.2}\left\{r_1'+\lambda_1\right\}-\frac{15}{16}\left\{\frac{3}{2}r_1'+\frac{1}{2}\lambda_1\right\}+\frac{5.3}{8.2}\left\{r_3'+\lambda_3\right\}\right\}e\cos(t+x)$$

[103]

$$+\frac{a}{a_{,}}\left\{+\frac{25.3}{8.2}\left\{r_1'+\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{2}r_1'+\frac{1}{2}\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{2}r_5'+\frac{1}{2}\lambda_5\right\}\right\}e_{,}\cos(t-z)$$

[104]

$$+\frac{a}{a_{,}}\left\{-\frac{5.3}{8.2}\left\{r_1'+\lambda_1\right\}+\frac{9}{8}\left\{\frac{3}{2}r_1'+\frac{1}{2}\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{2}r_5'-\frac{1}{2}\lambda_5\right\}\right\}e_{,}\cos(t+z)$$

[105]

$$+\frac{a}{a_{,}}\left\{-\frac{3}{16}\left\{\frac{3}{2}r_3'+\frac{1}{2}\lambda_3\right\}\right\}e^2\cos(t-2x)+\frac{15.3}{16.2}\frac{a}{a_{,}}\left\{r_3'+\lambda_3\right\}e^2\cos(t+2x)$$

[106] [107]

$$+\frac{a}{a_{,}}\left\{+\frac{3}{8}\left\{\frac{3}{2}r_3'+\frac{1}{2}\lambda_3\right\}-\frac{15}{16}\left\{\frac{3}{2}r_5'+\frac{1}{2}\lambda_5\right\}\right\}e\,e_{,}\cos(t-x-z)$$

[108]

$$+\frac{a}{a_{,}}\left\{-\frac{5.3}{8.2}\left\{r_3'+\lambda_3\right\}-\frac{3}{16}\left\{\frac{3}{2}r_5'-\frac{1}{2}\lambda_5\right\}\right\}e\,e_{,}\cos(t+x+z)$$

[109]

$$+\frac{a}{a_{,}}\left\{+\frac{9}{8}\left\{\frac{3}{2}r_3'+\frac{1}{2}\lambda_3\right\}-\frac{15}{16}\left\{\frac{3}{2}r_5'-\frac{1}{2}\lambda_5\right\}\right\}e\,e_{,}\cos(t-x+z)$$

[110]

$$+\frac{a}{a_{,}}\left\{+\frac{25.3}{8.2}\left\{r_3'+\lambda_3\right\}-\frac{3}{16}\left\{\frac{3}{2}r_5'+\frac{1}{2}\lambda_5\right\}\right\}e\,e_{,}\cos(t+x-z)$$

[111]

$$+\frac{a}{a_{,}}\frac{9}{8}\left\{\frac{3}{2}r_5'+\frac{1}{2}\lambda_5\right\}e_{,}^2\cos(t-2z)+\frac{a}{a_{,}}\frac{3}{8}\left\{\frac{3}{2}r_5'-\frac{1}{2}\lambda_5\right\}e_{,}^2\cos(t+2z)$$

[112] [113]

$$+\frac{a}{a_{,}}\left\{+\frac{3}{8}\left\{\frac{3}{2}r_1'-\frac{1}{2}\lambda_1\right\}-\frac{3}{16}\left\{\frac{3}{2}r_3'-\frac{1}{2}\lambda_3\right\}+\frac{25.3}{8.2}\left\{r_5'-\lambda_5\right\}\right.$$

$$\left.-\frac{5.3}{8.2}\left\{r_5'+\lambda_5\right\}\right\}\cos 3t$$

[116]

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* In this development of δR the terms multiplied by $\frac{a^3}{a_{,}^4}\gamma^2 s_{147}$ are neglected.

Development of δR.

$$+ \frac{a}{a_{,}} \left\{ - \frac{15}{16} \left\{ \frac{3}{2} r_1' - \frac{1}{2} \lambda_1 \right\} + \frac{3}{8} \left\{ \frac{3}{2} r_3' - \frac{1}{2} \lambda_3 \right\} \right\} e \cos (3t - x)$$

[117]

$$- \frac{a}{a_{,}} \frac{3}{16} \left\{ \frac{3}{2} r_1' - \frac{1}{2} \lambda_1 \right\} e \cos (3t + x)$$

[118]

$$+ \frac{a}{a_{,}} \left\{ + \frac{9}{8} \left\{ \frac{3}{2} r_1' - \frac{1}{2} \lambda_1 \right\} + \frac{5.3}{8.2} \left\{ r_1' + \lambda_1 \right\} \right\} e_{,} \cos (3t - z)$$

[119]

$$+ \frac{a}{a_{,}} \left\{ + \frac{3}{8} \left\{ \frac{3}{2} r_1' - \frac{1}{2} \lambda_1 \right\} + \frac{5.3}{8.2} \left\{ r_3' - \lambda_3 \right\} \right\} e_{,} \cos (3t + z)$$

[120]

$$- \frac{a}{a_{,}} \frac{15}{16} \left\{ \frac{3}{2} r_3' - \frac{1}{2} \lambda_3 \right\} e^2 \cos (3t - 2x)$$

[121]

$$+ \frac{a}{a_{,}} \left\{ + \frac{9}{8} \left\{ \frac{3}{2} r_3' - \frac{1}{2} \lambda_3 \right\} - \frac{45.3}{16.2} \left\{ r_5' + \lambda_5 \right\} \right\} e e_{,} \cos (3t - x - z)$$

[123]

$$+ \frac{a}{a_{,}} \frac{15.3}{16.2} \left\{ r_5' - \lambda_5 \right\} e e_{,} \cos (3t + x + z)$$

[124]

$$+ \frac{a}{a_{,}} \left\{ + \frac{3}{8} \left\{ \frac{3}{2} r_3' - \frac{1}{2} \lambda_3 \right\} - \frac{45.3}{16.2} \left\{ r_5 - \lambda_5 \right\} \right\} e e_{,} \cos (3t - x + z)$$

[125]

$$+ \frac{a}{a_{,}} \frac{15.3}{16.2} \left\{ r_5' + \lambda_5 \right\} e e_{,} \cos (3t + x - z) + \frac{25.3}{8.2} \left\{ r_5' + \lambda_5 \right\} e_{,}^2 \cos (3t - 2z)$$

[126] [127]

$$- \frac{5.3}{8.2} \left\{ r_5' - \lambda_5 \right\} e_{,}^2 \cos (3t + 2z)$$

[128]

$$+ \left\{ + \frac{20}{27} \left\{ r_1' - \lambda_1 \right\} + \frac{20}{27} e^2 \left\{ r_3' - \lambda_3 \right\} - \frac{10}{27} \gamma^2 s_{147} \right\} \cos 4t$$

[131]

$$+ \left\{ - \frac{38}{17} \left\{ r_1' - \lambda_1 \right\} + \frac{20}{27} \left\{ r_3' - \lambda_3 \right\} + \frac{9}{8} \gamma^2 s_{147} \right\} e \cos (4t - x)$$

[132]

$$+ \left\{ + \frac{20}{27} \left\{ r_1' - \lambda_1 \right\} + \frac{20}{27} e^2 \left\{ r_3' - \lambda_3 \right\} + \frac{20}{27} \left\{ r_4' - \lambda_4 \right\} - \frac{3}{8} \gamma^2 s_{147} \right\} e \cos (4t + x)$$

[133]

Development of δ *R*.

$$+\left\{+\frac{70}{27}\left\{r_1'-\lambda_1\right\}+\frac{83}{32}e^2\left\{r_3'-\lambda_3\right\}-\frac{21}{16}\gamma^2 s_{147}\right\}e_{,}\cos(4t-z)$$

[134]

$$+\left\{-\frac{10}{27}\left\{r_1'-\lambda_1\right\}-\frac{10}{27}\left\{r_3'-\lambda_3\right\}-\frac{3}{16}\gamma^2 s_{147}\right\}e_{,}\cos(4t+z)$$

[135]

$$+\left\{+\frac{28}{15}\left\{r_1'-\lambda_1\right\}-\frac{38}{17}\left\{r_3'-\lambda_3\right\}-\frac{15}{16}\gamma^2 s_{147}\right\}e^2\cos(4t-2x)$$

[136]

$$+\left\{+\frac{20}{27}r_1'+\frac{20}{27}\left\{r_4'-\lambda_4\right\}-\frac{3}{8}s_{147}\right\}e^2\cos(4t+2x)$$

[137]

$$+\left\{-\frac{180}{23}\left\{r_1'-\lambda_1\right\}+\frac{70}{27}\left\{r_3'-\lambda_3\right\}+\frac{63}{16}\gamma^2 s_{147}\right\}e\,e_{,}\cos(4t-x-z)$$

[138]

$$+\left\{-\frac{10}{27}\left\{r_1'-\lambda_1\right\}-\frac{10}{27}\left\{r_4'-\lambda_4\right\}+\frac{3}{16}\gamma^2 s_{147}\right\}e\,e_{,}\cos(4t+x+z)$$

[139]

$$+\left\{+\frac{66}{59}\left\{r_1'-\lambda_1\right\}-\frac{10}{27}\left\{r_3'-\lambda_3\right\}-\frac{9}{16}\gamma^2 s_{147}\right\}e\,e_{,}\cos(4t-x+z)$$

[140]

$$+\left\{+\frac{83}{32}\left\{r_1'-\lambda_1\right\}+\frac{70}{27}\left\{r_4'-\lambda_4\right\}-\frac{21}{16}\gamma^2 s_{147}\right\}e\,e_{,}\cos(4t+x-z)$$

[141]

$$+\left\{+\frac{233}{37}\left\{r_1'-\lambda_1\right\}-\frac{51}{16}\gamma^2 s_{147}\right\}e_{,}^2\cos(4t-2z)$$

[142]

$\delta\,.\,\mathrm{d}R =$ the differential of δR, supposing only $n\,t$ variable

$$+\frac{m^* m_{,} a^2}{a_{,}^3}\left\{\left\{+\frac{2\,.\,68}{137}r_1'-\frac{2\,.\,38}{77}e^2 r_3'-\frac{2\,.\,38}{77}e^2 r_4'-\frac{70}{27}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{10}{27}e_{,}^2\left\{r_5'+\lambda_5\right\}\right.\right.$$

$$\left.+\frac{3\,.\,32}{43}e_{,}^2 r_6'+\frac{32}{43}e_{,}^2 r_7'-\frac{2\,.\,102}{137}\gamma^2 s_{147}\right\}\sin 2t$$

[1]

$$+\left\{-\frac{2\,.\,20}{27}\left\{r_1'+\lambda_1\right\}-\frac{2\,.\,38}{17}\left\{r_1'+\lambda_1\right\}+\frac{2\,.\,28}{15}e^2\left\{r_3'+\lambda_3\right\}\right.$$

$$-\frac{2\,.\,20}{27}\left\{r_3'+\lambda_3\right\}+\frac{2\,.\,20}{27}\left\{r_4'+\lambda_4\right\}+\frac{20}{27}e_{,}^2 r_5'-\frac{20}{27}e_{,}^2 r_5'-\frac{2\,.\,3}{8}\gamma^2 s_{147}$$

$$\left.-\frac{2\,.\,9}{8}\gamma^2 s_{147}\right\}e\sin x$$

[2]

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* $m = \frac{n_{,}}{n}$ as in the notation of M. Damoiseau.

Development of $\delta\, d\, R$.

$$+\left\{-\frac{2.38}{77}r_1'+\frac{2.68}{137}r_3'+\frac{180}{23}e_{,}^2\left\{r_5'-\lambda_5\right\}+\frac{66}{59}e_{,}^2\left\{r_5'+\lambda_5\right\}\right.$$

$$\left.-\frac{2.3}{4}\gamma^2 s_{147}\right\}e\sin(2t-x)$$

[3]

$$+\left\{-\frac{2.38}{77}r_1'-\frac{2.10}{81}e^2r_3'+\frac{2.68}{137}r_4'-\frac{83}{32}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{10}{27}e_{,}^2\left\{r_5'+\lambda_5\right\}\right.$$

$$\left.+\frac{2.3}{4}\gamma^2 s_{147}\right\}e\sin(2t+x)$$

[4]

$$+\left\{+\frac{2.10}{27}\left\{r_1'+\lambda_1\right\}+\frac{2.70}{27}\left\{r_1'+\lambda_1\right\}-\frac{2.66}{59}e^2\left\{r_3'+\lambda_3\right\}-\frac{2.180}{23}e^2\left\{r_3'+\lambda_3\right\}\right.$$

$$\left.-\frac{68}{137}r_5'+\frac{67}{60}e_{,}^2r_5'-\frac{2.3}{16}\gamma^2 s_{147}+\frac{2.21}{16}\gamma^2 s_{147}\right\}e_{,}\sin z$$

[5]

$$+\left\{+\frac{2.32}{43}r_1'-\frac{2.20}{27}e^2r_3'-\frac{233}{37}e_{,}^2\left\{r_5'-\lambda_5\right\}+\frac{20}{27}\left\{r_5'+\lambda_5\right\}+\frac{3.68}{137}r_6'\right.$$

$$\left.+\frac{2.9}{8}\gamma^2 s_{147}\right\}e_{,}\sin(2t-z)$$

[6]

$$+\left\{+\frac{2.32}{43}r_1'-\frac{2.20}{27}e^2r_3'-\frac{20}{27}\left\{r_5'-\lambda_5\right\}+\frac{68}{137}r_7'-\frac{2.9}{8}\gamma^2 s_{147}\right\}e_{,}\sin(2t+z)$$

[7]

$$+\left\{-\frac{2.20}{27}\left\{r_1'+\lambda_1\right\}+\frac{2.28}{15}\left\{r_1'+\lambda_1\right\}-\frac{2.7}{32}e^2\left\{r_3'+\lambda_3\right\}-\frac{2.20}{27}\left\{r_3'+\lambda_3\right\}\right.$$

$$-\frac{2.38}{17}\left\{r_4'+\lambda_4\right\}+\frac{3}{16}e_{,}^2r_5'-\frac{3}{16}e_{,}^2r_5-\frac{2.20}{27}\left\{r_9'+\lambda_9\right\}+\frac{2.20}{27}\left\{r_{10}+\lambda_{10}\right\}$$

$$\left.-\frac{2.3}{8}\gamma^2 s_{147}+\frac{2.15}{16}\gamma^2 s_{147}\right\}e^2\sin 2x$$

[8]

$$+\left\{-\frac{2.10}{81}r_1'-\frac{2.38}{77}r_3'-\frac{105}{16}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{15}{16}e_{,}^2\left\{r_5'+\lambda_5\right\}+\frac{2.68}{137}r_9'\right.$$

$$\left.-\frac{2.3}{16}\gamma^2 s_{147}\right\}e^2\sin(2t-2x)$$

[9]

$$+\left\{-\frac{2.10}{81}r_1'-\frac{2}{16}e^2r_3'-\frac{2.38}{77}r_4'-\frac{21}{8}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{3}{8}e_{,}^2\left\{r_5'+\lambda_5\right\}+\frac{2.68}{137}r_{10}'\right.$$

Development of $\delta\, d R$.

$$+ \frac{2.3}{16} \gamma^2 s_{147} \Big\} e^2 \sin (2t + 2x)$$

[10]

$$+ \Big\{ + \frac{2.10}{27} \Big\{ r_1' + \lambda_1 \Big\} - \frac{2.180}{23} \Big\{ r_1' + \lambda_1 \Big\} + \frac{2.105}{16} e^2 \Big\{ r_3' + \lambda_3 \Big\} + \frac{2.10}{27} \Big\{ r_3' + \lambda_3 \Big\}$$

$$+ \frac{2.70}{27} \Big\{ r_4' + \lambda_4 \Big\} + \frac{38}{77} r_5' - \frac{9}{8} e_{,}^2 r_5' - \frac{3.20}{27} \Big\{ r_6' + \lambda_6 \Big\} - \frac{38}{17} \Big\{ r_7 + \lambda_7 \Big\}$$

$$- \frac{2.66}{59} e^2 \Big\{ r_9' + \lambda_9 \Big\} + \frac{2.83}{32} e^2 \Big\{ r_{10}' + \lambda_{10} \Big\} - \frac{68}{137} r_{11}' - \frac{3.20}{27} \Big\{ r_{12}' + \lambda_{12} \Big\}$$

$$+ \frac{3}{8} \gamma^2 s_{147} - \frac{2.63}{16} \gamma^2 s_{147} \Big\} e e_{,} \sin (x + z)$$

[11]

$$+ \Big\{ - \frac{2.20}{27} r_1' + \frac{2.32}{43} r_3' + \frac{153}{8} e_{,}^2 \Big\{ r_5' - \lambda_5 \Big\} - \frac{38}{17} \Big\{ r_5' + \lambda_5 \Big\} - \frac{3.38}{77} r_6'$$

$$+ \frac{20}{27} \Big\{ r_{11}' + \lambda_{11} \Big\} + \frac{3.68}{137} r_{12}' - \frac{9}{4} \gamma^2 s_{147} \Big\} e e_{,} \sin (2t - x - z)$$

[12]

$$+ \Big\{ - \frac{2.20}{27} r_1' - \frac{2.3}{16} e^2 \Big\{ r_3' - \lambda_3 \Big\} + \frac{2.32}{43} r_4' - \frac{20}{27} \Big\{ r_5' - \lambda_5 \Big\} - \frac{38}{77} r_7'$$

$$- \frac{20}{27} \Big\{ r_{11}' - \lambda_{11} \Big\} + \frac{68}{137} r_{13}' + \frac{9}{4} \gamma^2 s_{147} \Big\} e e_{,} \sin (2t + x + z)$$

[13]

$$+ \Big\{ - \frac{2.83}{32} \Big\{ r_1' + \lambda_1 \Big\} + \frac{2.66}{59} \Big\{ r_1' + \lambda_1 \Big\} - \frac{2.15}{16} e^2 \Big\{ r_3' + \lambda_3 \Big\} - \frac{2.70}{27} \Big\{ r_3' + \lambda_3 \Big\}$$

$$- \frac{2.10}{27} \Big\{ r_4' + \lambda_4 \Big\} + \frac{9}{8} e_{,}^2 r_5' - \frac{38}{77} e_{,}^2 r_5' - \frac{3.38}{77} \Big\{ r_6' + \lambda_6 \Big\} - \frac{20}{27} \Big\{ r_7' + \lambda_7 \Big\}$$

$$+ \frac{68}{137} r_{14}' - \frac{2.21}{16} \gamma^2 s_{147} + \frac{2.9}{16} \gamma^2 s_{147} \Big\} e e_{,} \sin (x - z)$$

[14]

$$+ \Big\{ - \frac{2.20}{27} r_1' + \frac{2.32}{43} r_3' + \frac{38}{17} \Big\{ r_5' - \lambda_5 \Big\} - \frac{20}{27} \Big\{ r_{14}' + \lambda_{14} \Big\} + \frac{68}{137} r_{15}'$$

$$- \frac{2.9}{8} \gamma^2 s_{147} \Big\} e e_{,} \sin (2t - x + z)$$

[15]

$$+ \Big\{ - \frac{2.20}{27} r_1' - \frac{2.3}{16} e^2 r_3' + \frac{2.32}{43} r_4' - \frac{51}{8} e_{,}^2 \Big\{ r_5' - \lambda_5 \Big\} + \frac{20}{27} \Big\{ r_5' + \lambda_5 \Big\} - \frac{3.38}{77} r_6'$$

Development of δ d R.

$$+ \frac{20}{27}\left\{r_{14}' - \lambda_{14}\right\} + \frac{3.68}{137} r_{16}' + \frac{2.9}{8}\gamma^2 s_{147}\biggr\} e\, e_{,} \sin(2t + x - z)$$

[16]

$$+ \biggl\{ + \frac{2.233}{37}\left\{r_1' + \lambda_1\right\} - \frac{2.153}{8} e^2 \left\{r_3' + \lambda_3\right\} - \frac{32}{43} r_5' + \frac{53}{32} e_{,}^2 r_5' + \frac{3.10}{27}\left\{r_6' + \lambda_6\right\}$$

$$+ \frac{70}{27}\left\{r_7' + \lambda_7\right\} + \frac{2.51}{16}\gamma^2 s_{147}\biggr\} e_{,}^2 \sin 2z$$

[17]

$$+ \biggl\{ + \frac{2.67}{60} r_1' - \frac{2.9}{8} e^2 r_3' - \frac{845}{64} e_{,}^2 \left\{r_5' - \lambda_5\right\} + \frac{70}{27}\left\{r_5' + \lambda_5\right\} + \frac{3.32}{43} r_6'$$

$$+ \frac{2.20}{27}\left\{r_{17}' + \lambda_{17}\right\} + \frac{4.68}{137} r_{18}' + \frac{2.27}{16}\gamma^2 s_{147}\biggr\} e_{,}^2 \sin(2t - 2z)$$

[18]

$$+ \biggl\{ + \frac{2.67}{60} r_1' - \frac{2.9}{8} e^2 r_3' + \frac{10}{27}\left\{r_5' - \lambda_5\right\} + \frac{e_{,}^2}{64}\left\{r_5 + \lambda_5\right\} + \frac{32}{43} r_7'$$

$$- \frac{2.20}{17}\left\{r_{17}' - \lambda_{17}\right\} - \frac{2.27}{16}\gamma^2 s_{147}\biggr\} e_{,}^2 \sin(2t + 2z)$$

[19]

$$+ \biggl\{ + \frac{2.26}{69}\left\{r_1' + \lambda_1\right\} - \frac{2.3}{8} e^2 \left\{r_3' + \lambda_3\right\} + \frac{9}{16} e_{,}^2 r_5' - \frac{9}{16} e_{,}^2 r_5' + \frac{20}{27} s_{147}\biggr\} \gamma^2 \cos 2y$$

[62]

$$+ \biggl\{ + \frac{2.16}{43} r_1' - \frac{2.9}{8} e^2 r_3' - \frac{21}{16} e_{,}^2 \left\{r_5' + \lambda_5\right\} - \frac{3}{16} e_{,}^2 \left\{r_5' + \lambda_5\right\}$$

$$- \frac{102}{137} s_{147}\biggr\} \gamma^2 \cos(2t - 2y)$$

[63]

$$+ \biggl\{ + \frac{2.16}{43} r_1' + \frac{2.3}{8} e^2 r_3'\biggr\} \gamma^2 \cos(2t + 2y)$$

[64]

$$+ \biggl\{ + \frac{2.3}{8}\left\{r_1' + \lambda_1\right\} - \frac{2.26}{69}\left\{r_3' + \lambda_3\right\} + \frac{9}{4} s_{147}\biggr\} \gamma^2 e \cos(x - 2y)$$

[65]

$$+ \biggl\{ - \frac{2.3}{8}\left\{r_1' + \lambda_1\right\} + \frac{3}{4} s_{147}\biggr\} \gamma^2 e \cos(x + 2y)$$

[66]

$$+ \biggl\{ + \frac{2.3}{8} r_1' + \frac{2.16}{43} r_3' + \frac{3}{2} s_{147}\biggr\} \gamma^2 e \cos(2t - x - 2y)$$

[67]

Development of $\delta\, d\, R$.

$$+\left\{-\frac{2\,.\,9}{8}r_1{}'+\frac{2\,.\,16}{43}r_3{}'\right\}\gamma^2 e\cos(2t-x+2y)$$

[68]

$$+\left\{-\frac{2\,.\,9}{8}r_1{}'-\frac{3}{2}s_{147}\right\}\gamma^2 e\cos(2t+x-2y)+\frac{2\,.\,3}{8}r_1{}'\gamma^2 e\cos(2t+x+2y)$$

[69] [70]

$$+\left\{+\frac{2\,.\,3}{16}\left\{r_1{}'+\lambda_1\right\}-\frac{16}{43}r_5{}'-\frac{21}{8}s_{147}\right\}\gamma^2 e\cos(z-2y)$$

[71]

$$+\left\{+\frac{2\,.\,21}{16}\left\{r_1{}'+\lambda_1\right\}-\frac{16}{43}r_5{}'+\frac{3}{8}s_{147}\right\}\gamma^2 e_{,}\cos(z+2y)$$

[72]

$$+\left\{+\frac{2\,.\,9}{16}r_1{}'+\frac{26}{69}r_5{}'-\frac{9}{4}s_{147}\right\}\gamma^2 e_{,}\cos(2t-z-2y)+\frac{2\,.\,9}{16}r_1{}'\gamma^2 e_{,}\cos(2t-z+2y)$$

[73] [74]

$$+\left\{+\frac{2\,.\,9}{16}r_1{}'-\frac{26}{69}\left\{r_5{}'+\lambda_5\right\}+\frac{9}{4}s_{147}\right\}\gamma^2 e_{,}\cos(2t+z-2y)$$

[75]

$$+\frac{2\,.\,9}{16}r_1\gamma^2 e_{,}\cos(2t+z+2y)$$

[76]

$$+\frac{a}{a_{,}}\left\{-\frac{2\,.\,5\,.\,3}{8\,.\,2}\left\{r_1{}'+\lambda_1\right\}+\frac{2\,.\,3}{8}\left\{\frac{3}{2}r_1{}'+\frac{1}{2}\lambda_1\right\}+\frac{2\,.\,45\,.\,3}{16\,.\,2}e^2\left\{r_3{}'+\lambda_3\right\}\right.$$

$$-\frac{2\,.\,15}{16}e^2\left\{\frac{3}{9}r_3{}'+\frac{1}{2}\lambda_3\right\}-\frac{9}{8}e_{,}^2\left\{\frac{3}{2}r_5{}'+\frac{1}{2}\lambda_5\right\}$$

$$\left.+\frac{3}{8}e_{,}^2\left\{\frac{3}{2}r_5{}'+\frac{1}{2}\lambda_5\right\}\right\}\cos t$$

[101]

$$+\frac{a}{a_{,}}\left\{+\frac{2\,.\,45\,.\,3}{16\,.\,2}\left\{r_1{}'+\lambda_1\right\}-\frac{2\,.\,3}{16}\left\{\frac{3}{2}r_1{}'+\frac{1}{2}\lambda_1\right\}\right.$$

$$\left.+\frac{2\,.\,3}{8}\left\{\frac{3}{2}r_3{}'+\frac{1}{2}\lambda_3\right\}\right\}e\cos(t-x)$$

[102]

$$+\frac{a}{a_{,}}\left\{-\frac{2\,.\,15\,.\,3}{16\,.\,2}\left\{r_1{}'+\lambda_1\right\}-\frac{2\,.\,15}{16}\left\{\frac{3}{2}r_1{}'+\frac{1}{2}\lambda_1\right\}\right.$$

$$\left.-\frac{2\,.\,5\,.\,3}{8\,.\,2}\left\{r_3{}'+\lambda_3\right\}\right\}e\cos(t+x)$$

[103]

$$+\frac{a}{a_{,}}\left\{-\frac{2\,.\,25\,.\,3}{8\,.\,2}\left\{r_1{}'+\lambda_1\right\}+\frac{2\,.\,3}{8}\left\{\frac{3}{2}r_1{}'+\frac{1}{2}\lambda_1\right\}\right.$$

Development of $\delta\, \mathrm{d} R$,

$$+ \frac{3}{8}\left\{\frac{3}{2} r_5' + \frac{1}{2}\lambda_5\right\}\Bigg] e_{,} \cos(t - z)$$

[104]

$$+ \frac{a}{a_{,}}\Bigg[+ \frac{2 . 5 . 3}{8 . 2}\left\{r_1' + \lambda_1\right\} + \frac{2 . 9}{8}\left\{\frac{3}{2} r_1' + \frac{1}{2}\lambda_1\right\}$$

$$- \frac{3}{8}\left\{\frac{3}{2} r_5' - \frac{1}{2}\lambda_5\right\}\Bigg] e_{,} \cos(t + z)$$

[105]

$$+ \frac{a}{a_{,}}\Bigg[+ \frac{2 . 9}{8}\left\{\frac{3}{2} r_3' + \frac{1}{2}\lambda_3\right\} + \frac{15}{16}\left\{\frac{3}{2} r_5' - \frac{1}{2}\lambda_5\right\}\Bigg] e\, e_{,} \cos(t - x + z)$$

[110]

$$+ \frac{a}{a_{,}}\Bigg[+ \frac{2 . 3}{8}\left\{\frac{3}{2} r_1' - \frac{1}{2}\lambda_1\right\} - \frac{2 . 3}{16} e^2 \left\{\frac{3}{2} r_3' - \frac{1}{2}\lambda_3\right\} - \frac{25 . 3}{8 . 2} e_{,}^2 \left\{r_5' - \lambda_5\right\}$$

$$- \frac{5 . 3}{8 . 2} e_{,}^2 \left\{r_5' + \lambda_5\right\}\Bigg] \cos 3t$$

[116]

$$+ \frac{a}{a_{,}}\Bigg[- \frac{2 . 15}{16}\left\{\frac{3}{2} r_1' - \frac{1}{2}\lambda_1\right\} + \frac{2 . 3}{8}\left\{\frac{3}{2} r_3 - \frac{1}{2}\lambda_3\right\}\Bigg] e \cos(3t - x)$$

[117]

$$- \frac{a}{a_{,}} \frac{2 . 3}{16}\left\{\frac{3}{2} r_1 - \frac{1}{2}\lambda_1\right\} e \cos(3t + x)$$

[118]

$$+ \frac{a}{a_{,}}\Bigg[+ \frac{2 . 9}{8}\left\{\frac{3}{2} r_1' - \frac{1}{2}\lambda_1\right\} + \frac{5 . 3}{8 . 2}\left\{r_5' + \lambda_5\right\}\Bigg] e_{,} \cos(3t - z)$$

[119]

$$+ \frac{a}{a_{,}}\Bigg[+ \frac{2 . 3}{8}\left\{\frac{3}{2} r_1 - \frac{1}{2}\lambda_1\right\} - \frac{5 . 3}{8 . 2}\left\{r_5' - \lambda_5\right\}\Bigg] e_{,} \cos(3t + z)$$

[120]

$$+ \Bigg[+ \frac{2 . 20}{27}\left\{r_1' - \lambda_1\right\} + \frac{2 . 20}{27} e^2 \left\{r_3' - \lambda_3\right\} - \frac{20}{27}\gamma^2 s_{147}\Bigg] \cos 4t$$

[131]

$$+ \Bigg[- \frac{2 . 38}{17}\left\{r_1' - \lambda_1\right\} + \frac{2 . 20}{27}\left\{r_3' - \lambda_3\right\} + \frac{9}{4}\gamma^2 s_{147}\Bigg] e \cos(4t - x)$$

[132]

$$+ \Bigg[+ \frac{2 . 20}{27}\left\{r_1' - \lambda_1\right\} + \frac{2 . 20}{27}\left\{r_3' - \lambda_3\right\} + \frac{2 . 20}{27}\left\{r_4' - \lambda_4\right\}$$

$$- \frac{3}{4}\gamma^2 s_{147}\Bigg] e \cos(4t + x)$$

[133]

$$+ \Bigg[+ \frac{2 . 70}{27}\left\{r_1' - \lambda_1\right\} + \frac{2 . 83}{32} e^2 \left\{r_3' - \lambda_3\right\} - \frac{21}{8}\gamma^2 s_{147}\Bigg] e_{,} \cos(4t - z)$$

[134]

Development of δ d R.

$$+\left\{-\frac{2.10}{27}\left\{r_1'-\lambda_1\right\}-\frac{2.10}{27}\left\{r_3'-\lambda_3\right\}-\frac{3}{8}\gamma^2 s_{147}\right\}e_{\prime}\cos(4t+z)$$
[135]

$$+\left\{+\frac{2.28}{15}\left\{r_1'-\lambda_1\right\}-\frac{2.38}{17}\left\{r_3'-\lambda_3\right\}-\frac{15}{8}\gamma^2 s_{147}\right\}e^2\cos(4t-2x)$$
[136]

$$+\left\{+\frac{2.20}{27}r_1'+\frac{2.20}{27}\left\{r_4'-\lambda_4\right\}-\frac{3}{4}\gamma^2 s_{147}\right\}e^2\cos(4t+2x)$$
[137]

$$+\left\{-\frac{2.180}{23}\left\{r_1'-\lambda_1\right\}+\frac{2.70}{27}\left\{r_3'-\lambda_3\right\}-\frac{63}{8}\gamma^2 s_{147}\right\}e\,e_{\prime}\cos(4t-x-z)$$
[138]

$$+\left\{-\frac{2.10}{27}\left\{r_1'-\lambda_1\right\}-\frac{2.10}{27}\left\{r_4'-\lambda_4\right\}-\frac{3}{8}\gamma^2 s_{147}\right\}e\,e_{\prime}\cos(4t+x+z)$$
[139]

$$+\left\{+\frac{2.66}{59}\left\{r_1'-\lambda_1\right\}-\frac{2.10}{27}\left\{r_3'-\lambda_3\right\}-\frac{9}{8}\gamma^2 s_{147}\right\}e\,e_{\prime}\cos(4t-x+z)$$
[140]

$$+\left\{+\frac{2.83}{32}\left\{r_1'-\lambda_1\right\}+\frac{70}{27}\left\{r_4'-\lambda_4\right\}-\frac{21}{8}\gamma^2 s_{147}\right\}e\,e_{\prime}\cos(4t+x-z)$$
[141]

$$+\left\{+\frac{2.233}{37}\left\{r_1-\lambda_1\right\}-\frac{51}{8}\gamma^2 s_{147}\right\}e_{\prime}^2\cos(4t-2z)$$
[142]

$\delta\,.\,r\left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)$ and $\delta\int \mathrm{d}R$ may be obtained immediately from the preceding developments.

Developments required for the integration of the equation

$$\frac{\mathrm{d}\grave{\lambda}}{\mathrm{d}t}=h\frac{(1+s^2)}{r^4}-\frac{(1+s^2)}{r^2}\int\left(\frac{\mathrm{d}R}{\mathrm{d}\grave{\lambda}}\right)\mathrm{d}t+\frac{(1+s^2)}{2r^2h}\left\{\int\left(\frac{\mathrm{d}R}{\mathrm{d}\grave{\lambda}}\right)\mathrm{d}t\right\}^2$$

$$\mathrm{d}\,.\,\frac{\left(\frac{\mathrm{d}R}{\mathrm{d}\grave{\lambda}}\right)}{\mathrm{d}s}=-\frac{2.2.3}{4}\frac{r^2}{r_{\prime}^3}\sin(2\grave{\lambda}-2\lambda_{\prime})\,s=-2\left(\frac{\mathrm{d}R}{\mathrm{d}\grave{\lambda}}\right)s$$

$$=\frac{2m_{\prime}a^2}{a_{\prime}^3}\left\{-\frac{20}{27}\gamma\cos(2t-y)+\frac{20}{27}\gamma\cos(2t+y)+\frac{38}{17}\gamma e\cos(2t-x-y)\right.$$
[147] [148] [151]

$$-\frac{38}{17}\gamma e\cos(2t-x+y)-\frac{20}{27}\gamma e\cos(2t+x-y)+\frac{20}{27}\gamma e\cos(2t+x+y)$$
[152] [153] [154]

$$-\frac{70}{27}\gamma e_{\prime}\cos(2t-z-y)+\frac{70}{27}\gamma e_{\prime}\cos(2t-z+y)+\frac{10}{27}\gamma e_{\prime}\cos(2t+z-y)$$
[157] [158] [159]

$$-\frac{10}{27}\gamma e_{\prime}\cos(2t+z+y)-\frac{28}{15}\gamma e^2\cos(2t-2x-y)+\frac{28}{15}\gamma^2 e^2\cos(2t-2x+y)$$

[160] [163] [164]

$$-\frac{20}{27}\gamma e^2\cos(2t+2x-y)+\frac{20}{27}e^2\cos(2t+2x+y)$$

[165] [166]

$$+\frac{180}{23}\gamma e e_{\prime}\cos(2t-x-z-y)-\frac{180}{23}\gamma e e_{\prime}\cos(2t-x-z+y)$$

[169] [170]

$$+\frac{10}{27}\gamma e e_{\prime}\cos(2t+x+z-y)-\frac{10}{27}\gamma e e_{\prime}\cos(2t+x+z+y)$$

[171] [172]

$$-\frac{66}{59}\gamma e e_{\prime}\cos(2t-x+z-y)+\frac{66}{59}\gamma e e_{\prime}\cos(2t-x+z+y)$$

[175] [176]

$$-\frac{83}{32}\gamma e e_{\prime}\cos(2t+x-z-y)+\frac{83}{32}\gamma e e_{\prime}\cos(2t+x-z+y)$$

[177] [178]

$$-\frac{233}{37}e_{\prime}^2\cos(2t-2z-y)+\frac{233}{37}e_{\prime}^2\cos(2t-2z+y)$$

[181] [182]

$$\delta\left(\frac{dR}{d\lambda}\right)=\frac{2m_{\prime}a^2}{a_{\prime}^3}\Bigg\{-\frac{40}{27}r_0'+\frac{38}{17}e^2r_2'-\frac{20}{27}e^2r_2'-\frac{70}{27}e_{\prime}^2\Big\{r_5'-\lambda_5\Big\}+\frac{10}{27}e_{\prime}^2\Big\{r_5'+\lambda_5\Big\}$$

$$-\frac{28}{15}e^4\Big\{r_8-\lambda_8\Big\}-\frac{20}{27}e^4\Big\{r_8+\lambda_8\Big\}+\frac{180}{23}e^2e_{\prime}^2\Big\{r_{11}'-\lambda_{11}\Big\}+\frac{10}{27}e^2e_{\prime}^2\Big\{r_{11}'+\lambda_{11}\Big\}$$

$$-\frac{66}{59}e^2e_{\prime}^2\Big\{r_{14}'-\lambda_{14}\Big\}-\frac{83}{32}e^2e_{\prime}^2\Big\{r_{14}'+\lambda_{14}\Big\}\Bigg\}\sin 2t$$

[1]

$$+\Bigg\{-\frac{20}{27}\Big\{r_1'+\lambda_1\Big\}-\frac{38}{17}\Big\{r_1'+\lambda_1\Big\}+\frac{28}{15}e^2\Big\{r_3'+\lambda_3\Big\}-\frac{20}{27}\Big\{r_3'+\lambda_3\Big\}$$

$$+\frac{20}{27}\Big\{r_4'+\lambda_4\Big\}+\frac{19}{17}\gamma^2 s_{147}+\frac{10}{27}\gamma^2 s_{147}\Bigg\}e\cos x$$

[2]

$$+\Bigg\{+\frac{76}{17}r_0'-\frac{20}{27}r_2'+\frac{180}{23}e_{\prime}^2\Big\{r_5'-\lambda_5\Big\}-\frac{66}{59}e_{\prime}^2\Big\{r_5'+\lambda_5\Big\}\Bigg\}e\sin(2t-x)$$

[3]

$$+\Bigg\{-\frac{40}{27}r_0'-\frac{20}{27}r_2'-\frac{83}{32}e_{\prime}^2\Big\{r_5'-\lambda_5\Big\}+\frac{10}{27}e_{\prime}^2\Big\{r_5'+\lambda_5\Big\}\Bigg\}e\sin(2t+x)$$

[4]

Development of $\delta\left(\frac{dR}{d\lambda'}\right)$.

$$+\left\{+\frac{10}{27}\left\{r_1'+\lambda_1\right\}+\frac{70}{27}\left\{r_1'+\lambda_1\right\}-\frac{66}{59}e^2\left\{r_3'+\lambda_3\right\}-\frac{180}{23}e^2\left\{r_3'+\lambda_3\right\}\right.$$

$$\left.+\frac{5}{27}\gamma^2 s_{147}-\frac{35}{27}\gamma^2 s_{147}\right\}e_{,}\sin z$$

[5]

$$+\left\{-\frac{140}{27}r_0'-\frac{233}{37}e_{,}^2\left\{r_5'-\lambda_5\right\}-\frac{20}{27}\left\{r_5'+\lambda_5\right\}\right\}e_{,}\sin(2t-z)$$

[6]

$$+\left\{+\frac{20}{27}r_0'-\frac{20}{27}\left\{r_5'-\lambda_5\right\}\right\}e_{,}\sin(2t+z)$$

[7]

$$+\left\{-\frac{20}{27}\left\{r_1'+\lambda_1\right\}+\frac{28}{15}\left\{r_1'+\lambda_1\right\}-\frac{7}{32}e^2\left\{r_3'+\lambda_3\right\}-\frac{20}{27}\left\{r_3'+\lambda_3\right\}\right.$$

$$-\frac{38}{17}\left\{r_4'+\lambda_4\right\}-\frac{20}{27}\left\{r_9'+\lambda_9\right\}+\frac{20}{27}\left\{r_{10}'+\lambda_{10}\right\}+\frac{10}{27}\gamma^2 s_{147}$$

$$\left.-\frac{14}{15}\gamma^2 s_{147}\right\}e^2\sin 2x$$

[8]

$$+\left\{-\frac{56}{15}r_0'+\frac{38}{17}r_2'-\frac{105}{16}e_{,}^2\left\{r_5'-\lambda_5\right\}+\frac{15}{16}e_{,}^2\left\{r_5'+\lambda_5\right\}\right.$$

$$\left.-\frac{20}{27}\left\{r_8'+\lambda_8\right\}\right\}e^2\sin(2t-2x)$$

[9]

$$+\left\{-\frac{40}{27}r_0'-\frac{21}{8}e_{,}^2\left\{r_5'-\lambda_5\right\}+\frac{3}{8}e_{,}^2\left\{r_5'+\lambda_5\right\}-\frac{20}{27}\left\{r_8'+\lambda_8\right\}\right\}e^2\cos(2t+2x)$$

[10]

$$+\left\{+\frac{10}{27}\left\{r_1'+\lambda_1\right\}+\frac{180}{23}\left\{r_1'+\lambda_1\right\}+\frac{105}{16}e^2\left\{r_3'+\lambda_3\right\}+\frac{10}{27}\left\{r_3'+\lambda_3\right\}\right.$$

$$+\frac{70}{27}\left\{r_4'+\lambda_4\right\}-\frac{20}{27}\left\{r_6'+\lambda_6\right\}-\frac{38}{17}\left\{r_7'+\lambda_7\right\}-\frac{66}{59}e^2\left\{r_9'+\lambda_9\right\}$$

$$+\frac{83}{32}e^2\left\{r_{10}'+\lambda_{10}\right\}-\frac{20}{27}\left\{r_{12}'+\lambda_{12}\right\}-\frac{5}{27}\gamma^2 s_{147}$$

$$\left.+\frac{90}{23}\gamma^2 s_{147}\right\}e\,e_{,}\sin(x+z)$$

[11]

$$+\left\{-\frac{360}{23}r_0'-\frac{70}{27}r_2'+\frac{153}{8}e_{,}^2\left\{r_5'-\lambda_5\right\}+\frac{38}{17}\left\{r_5'+\lambda_5\right\}\right.$$

$$\left.-\frac{20}{27}\left\{r_{11}'+\lambda_{11}\right\}\right\}e\,e_{,}\sin(2t-x-z)$$

[12]

Development of $\delta\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)$.

$$\begin{aligned}&+\Bigg\{+\frac{20}{27}r_0+\frac{10}{27}r_2'+\frac{3}{16}e^2\Big\{r_3'-\lambda_3\Big\}-\frac{20}{27}\Big\{r_5'-\lambda_5\Big\}\\&\quad-\frac{20}{27}\Big\{r_{11}'-\lambda_{11}\Big\}\Bigg\}e\,e_{,}\sin(2t+x+z)\end{aligned}$$

[13]

$$\begin{aligned}&+\Bigg\{-\frac{83}{32}\Big\{r_1'+\lambda_1\Big\}+\frac{66}{59}\Big\{r_1'+\lambda_1\Big\}-\frac{15}{16}e^2\Big\{r_3'+\lambda_3\Big\}-\frac{70}{27}\Big\{r_3'+\lambda_3\Big\}\\&\quad-\frac{10}{27}\Big\{r_4'+\lambda_4\Big\}-\frac{38}{17}\Big\{r_6'+\lambda_6\Big\}-\frac{20}{27}\Big\{r_7'-\lambda_7\Big\}+\frac{180}{23}e^2\Big\{r_9'-\lambda_9\Big\}\\&\quad-\frac{10}{27}e^2\Big\{r_{10}'+\lambda_{10}\Big\}+\frac{28}{15}e^2\Big\{r_{12}'+\lambda_{12}\Big\}-\frac{20}{27}\Big\{r_{15}'+\lambda_{15}\Big\}+\frac{20}{27}\Big\{r_{16}'+\lambda_{16}\Big\}\\&\quad+\frac{83}{64}\gamma^2s_{147}-\frac{66}{59}\gamma^2s_{147}\Bigg\}e\,e_{,}\sin(x-z)\end{aligned}$$

[14]

$$+\Bigg\{-\frac{132}{59}r'_0+\frac{10}{27}r_2'+\frac{38}{17}\Big\{r_5'-\lambda_5\Big\}-\frac{20}{27}\Big\{r_{14}'+\lambda_{14}\Big\}\Bigg\}e\,e_{,}\sin(2t-x+z)$$

[15]

$$+\Bigg\{-\frac{83}{16}r'_0-\frac{51}{8}e_{,}^2\Big\{r_5'-\lambda_5\Big\}-\frac{20}{27}\Big\{r_5'+\lambda_5\Big\}-\frac{20}{27}\Big\{r_{14}'-\lambda_{14}\Big\}\Bigg\}e\,e_{,}\sin(2t+x-z)$$

[16]

$$\begin{aligned}&+\Bigg\{+\frac{233}{37}\Big\{r_1'+\lambda_1\Big\}-\frac{153}{8}e^2\Big\{r_3'+\lambda_3\Big\}+\frac{10}{27}\Big\{r_6'+\lambda_6\Big\}+\frac{70}{27}\Big\{r_7'+\lambda_7\Big\}\\&\quad-\frac{20}{27}\Big\{r_{18}'+\lambda_{18}\Big\}+\frac{20}{27}\Big\{r_{19}'+\lambda_{19}\Big\}-\frac{233}{74}\gamma^2s_{147}\Bigg\}e_{,}^2\sin 2z\end{aligned}$$

[17]

$$+\Bigg\{-\frac{466}{37}r'_0-\frac{845}{64}e_{,}^2\Big\{r_5'-\lambda_5\Big\}-\frac{70}{27}\Big\{r_5'+\lambda_5\Big\}-\frac{20}{27}\Big\{r_{17}'-\lambda_{17}\Big\}\Bigg\}e_{,}^2\sin(2t-2z)$$

[18]

$$+\Bigg\{+\frac{10}{27}\Big\{r_5'+\lambda_5\Big\}-\frac{e_{,}^2}{64}\Big\{r_5'+\lambda_5\Big\}-\frac{20}{27}\Big\{r_{17}'-\lambda_{17}\Big\}\Bigg\}e_{,}^2\sin(2t+2z)$$

[19]

$$+\Bigg\{+\frac{26}{69}\Big\{r_1'+\lambda_1\Big\}-\frac{3}{8}e^2\Big\{r_3'+\lambda_3\Big\}-\frac{10}{27}s_{147}\Bigg\}\gamma^2\sin 2y$$

[62]

$$+\Bigg\{-\frac{21}{16}e_{,}^2\Big\{r_5'-\lambda_5\Big\}+\frac{3}{16}e_{,}^2\Big\{r_5'+\lambda_5\Big\}\Bigg\}\gamma^2\sin(2t-2y)$$

[63]

$$+\Bigg\{+\frac{3}{8}\Big\{r_1'+\lambda_1\Big\}-\frac{26}{69}\Big\{r_3'+\lambda_3\Big\}-\frac{9}{8}s_{147}\Bigg\}\gamma^2e\sin(x-2y)$$

[65]

$$+\Bigg\{-\frac{3}{8}\Big\{r_1'+\lambda_1\Big\}-\frac{3}{8}s_{147}\Bigg\}\gamma^2e\sin(x+2y)$$

[66]

Development of $\delta\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)$.

$$+\left\{+\frac{3}{16}\left\{r_1'+\lambda_1\right\}+\frac{21}{16}s_{147}\right\}\gamma^2 e\sin(z-2y)$$

[71]

$$+\left\{+\frac{21}{16}\left\{r_1'+\lambda_1\right\}-\frac{3}{16}s_{147}\right\}\gamma^2 e_{\prime}\sin(z+2y)-\frac{26}{69}\left\{r_5'-\lambda_5\right\}\gamma^2 e_{\prime}\sin(2t-z-2y)$$

[72] [73]

$$-\frac{26}{69}\left\{r_5'+\lambda_5\right\}\gamma^2 e_{\prime}\sin(2t+z-2y)$$

[75]

$$+\left\{+\frac{3}{8}\left\{r_3'+\lambda_3\right\}+\frac{15}{16}s_{147}\right\}\gamma^2 e^2\sin(2x-2y)-\frac{3}{8}s_{147}\gamma^2 e^2\sin(2x+2y)$$

[77] [78]

$$+\left\{+\frac{3}{16}\left\{r_3'+\lambda_3\right\}-\frac{63}{16}s_{147}\right\}\gamma^2 e\,e_{\prime}\sin(x+z-2y)-\frac{3}{16}s_{147}\gamma^2 e\,e_{\prime}\sin(x+z+2y)$$

[83] [84]

$$+\frac{3}{8}\left\{r_5'+\lambda_5\right\}\gamma^2 e\,e_{\prime}\sin(2t-x-z-2y)$$

[85]

$$+\frac{3}{8}\left\{r_5'-\lambda_5\right\}\gamma^2 e\,e_{\prime}\sin(2t+x+z-2y)$$

[87]

$$+\left\{-\frac{21}{16}\left\{r_3'+\lambda_3\right\}+\frac{9}{16}s_{147}\right\}\gamma^2 e\,e_{\prime}\sin(x-z-2y)-\frac{21}{16}s_{147}\gamma^2 e\,e_{\prime}\sin(x-z+2y)$$

[89] [90]

$$+\frac{3}{8}\left\{r_5'-\lambda_5\right\}\gamma^2 e\,e_{\prime}\sin(2t-x+z-2y)$$

[91]

$$+\frac{3}{8}\left\{r_5'+\lambda_5\right\}\gamma^2 e\,e_{\prime}\sin(2t+x-z-2y)+\frac{51}{16}s_{147}\gamma^2 e_{\prime}^2\sin(2z-2y)$$

[93] [95]

$$+\frac{21}{16}\left\{r_5'+\lambda_5\right\}\gamma^2 e_{\prime}^2\sin(2t-2z-2y)+\frac{3}{16}\left\{r_5'-\lambda_5\right\}\gamma^2 e_{\prime}^2\sin(2t+2z-2y)$$

[97] [99]

$$+\frac{a}{a_{\prime}}\left\{-\frac{5.9}{8.4}\left\{r_1'+\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{4}r_1'+\frac{1}{4}\lambda_1\right\}+\frac{45.9}{16.4}e^2\left\{r_3'+\lambda_3\right\}\right.$$

$$-\frac{15}{16}e^2\left\{\frac{3}{4}r_3+\frac{1}{4}\lambda_3\right\}-\frac{9}{8}e_{\prime}^2\left\{\frac{3}{4}r_5'-\frac{1}{4}\lambda_5\right\}$$

$$\left.-\frac{3}{8}e_{\prime}^2\left\{\frac{3}{4}r_5'+\frac{1}{4}\lambda_5\right\}\right\}\sin t$$

[101]

$$+\frac{a}{a_{\prime}}\left\{+\frac{45.9}{16.4}\left\{r_1'+\lambda_1\right\}-\frac{3}{16}\left\{\frac{3}{4}r_1'+\frac{1}{4}\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{4}r_3'+\frac{1}{4}\lambda_3\right\}\right\}e\sin(t-x)$$

[102]

Development of $\delta\left(\frac{d R}{d \lambda}\right)$

$$+\frac{a}{a_{,}}\left\{-\frac{15.9}{16.4}\left\{r_1'+\lambda_1\right\}-\frac{15}{16}\left\{\frac{3}{4}r_1'+\frac{1}{4}\lambda_1\right\}-\frac{5.9}{8.4}\left\{r_3'+\lambda_3\right\}\right\}e\sin(t+x)$$

[103]

$$+\frac{a}{a_{,}}\left\{-\frac{25.9}{8.4}\left\{r_1'+\lambda_1\right\}+\frac{3}{8}\left\{\frac{3}{4}r_1'+\frac{1}{4}\lambda_1\right\}-\frac{3}{8}\left\{\frac{3}{4}r_5'+\frac{1}{4}\lambda_5\right\}\right\}e_{,}\sin(t-z)$$

[104]

$$+\frac{a}{a_{,}}\left\{+\frac{5.9}{8.4}\left\{r_1'+\lambda_1\right\}+\frac{9}{8}\left\{\frac{3}{4}r_1'+\frac{1}{4}\lambda_1\right\}-\frac{3}{8}\left\{\frac{3}{4}r_5'+\frac{1}{4}\lambda_5\right\}\right\}e_{,}\sin(t+z)$$

[105]

$$-\frac{a}{a_{,}}\frac{3}{16}\left\{\frac{3}{4}r_3'+\frac{1}{4}\lambda_3\right\}e^2\sin(t-2x)-\frac{a}{a_{,}}\frac{15.9}{16.4}\left\{r_3'+\lambda_3\right\}e^2\sin(t+2x)$$

[106] [107]

$$+\frac{a}{a_{,}}\left\{+\frac{3}{8}\left\{\frac{3}{4}r_3'+\frac{1}{4}\lambda_3\right\}+\frac{15}{16}\left\{\frac{3}{4}r_3'+\frac{1}{4}\lambda_5\right\}\right\}e\,e_{,}\sin(t-x-z)$$

[108]

$$+\frac{a}{a_{,}}\left\{+\frac{5.9}{8.4}\left\{r_3'+\lambda_3\right\}+\frac{3}{16}\left\{\frac{3}{4}r_5'-\frac{1}{4}\lambda_5\right\}\right\}e\,e_{,}\sin(t+x+z)$$

[109]

$$+\frac{a}{a_{,}}\left\{+\frac{9}{8}\left\{\frac{3}{4}r_3'+\frac{1}{4}\lambda_3\right\}+\frac{15}{16}\left\{\frac{3}{4}r_5'-\frac{1}{4}\lambda_5\right\}\right\}e\,e_{,}\sin(t-x+z)$$

[110]

$$+\frac{a}{a_{,}}\left\{-\frac{25.9}{8.4}\left\{r_3'+\lambda_3\right\}+\frac{3}{16}\Big\}\frac{3}{4}r_5'+\frac{1}{4}\lambda_5\Big\}\right\}e\,e_{,}\sin(t+x-z)$$

[111]

$$-\frac{a}{a_{,}}\frac{9}{8}\left\{\frac{3}{4}r_5'+\frac{1}{4}\lambda_5\right\}e_{,}^2\sin(t-2z)-\frac{a}{a_{,}}\frac{3}{8}\left\{\frac{3}{4}r_5'-\frac{1}{4}\lambda_5\right\}e_{,}^2\sin(t+2z)$$

[112] [113]

$$+\frac{a}{a_{,}}\left\{-\frac{3}{8}\left\{\frac{3}{4}r_1-\frac{1}{4}\lambda_1\right\}+\frac{3}{16}e^2\left\{\frac{3}{4}r_3'-\frac{1}{4}\lambda_3\right\}-\frac{25.9}{8.4}e_{,}^2\left\{r_5'-\lambda_5\right\}\right.$$
$$\left.+\frac{5.9}{8.4}e_{,}^2\left\{r_5'+\lambda_5\right\}\right\}\sin 3t$$

[116]

$$+\frac{a}{a_{,}}\left\{+\frac{15}{16}\left\{\frac{3}{4}r_1'-\frac{1}{4}\lambda_1\right\}-\frac{3}{8}\left\{\frac{3}{4}r_3'-\frac{1}{4}\lambda_3\right\}\right\}e\sin(3t-x)$$

[117]

$$+\frac{a}{a_{,}}\frac{3}{16}\left\{\frac{3}{4}r_1'-\frac{1}{4}\lambda_1\right\}e\sin(3t+x)$$

[118]

$$+\frac{a}{a_{,}}\left\{-\frac{9}{8}\left\{\frac{3}{4}r_1'-\frac{1}{4}\lambda_1\right\}-\frac{5.9}{8.4}\left\{r_5'+\lambda_5\right\}\right\}e_{,}\sin(3t-z)$$

[119]

Development of $\delta\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)$.

$$+\frac{a}{a_{\prime}}\left\{-\frac{3}{8}\left\{\frac{3}{4}r_1'-\frac{1}{4}\lambda_1\right\}-\frac{5\,.\,9}{8\,.\,4}\left\{r_5'-\lambda_5\right\}\right\}e_{\prime}\sin(3t+z)$$
[120]

$$+\frac{a}{a_{\prime}}\frac{15}{16}\left\{\frac{3}{4}r_3'-\frac{1}{4}\lambda_3\right\}e^2\sin(3t-2x)$$
[121]

$$+\frac{a}{a_{\prime}}\left\{-\frac{9}{8}\left\{\frac{3}{4}r_3'-\frac{1}{4}\lambda_3\right\}+\frac{45\,.\,9}{16\,.\,4}\left\{r_5'+\lambda_5\right\}\right\}ee_{\prime}\sin(3t-x-z)$$
[123]

$$-\frac{a}{a_{\prime}}\frac{15\,.\,9}{16\,.\,4}\left\{r_5'-\lambda_5\right\}ee_{\prime}\sin(3t+x+z)$$
[124]

$$+\frac{a}{a_{\prime}}\left\{-\frac{3}{8}\left\{\frac{3}{4}r_3'-\frac{1}{4}\lambda_3\right\}+\frac{45\,.\,9}{16\,.\,4}\left\{r_5'-\lambda_5\right\}\right\}ee_{\prime}\sin(3t-x+z)$$
[125]

$$-\frac{a}{a_{\prime}}\frac{15\,.\,9}{16\,.\,4}\left\{r_5'+\lambda_5\right\}ee_{\prime}\sin(3t+x-z)-\frac{a}{a_{\prime}}\frac{25\,.\,9}{8\,.\,4}\left\{r_5'+\lambda_5\right\}e_{\prime}^2\sin(3t-2z)$$
[126] [127]

$$+\frac{a}{a_{\prime}}\frac{5\,.\,9}{8\,.\,4}\left\{r_5'-\lambda_5\right\}e_{\prime}^2\sin(3t+2z)$$
[128]

$$+\left\{-\frac{20}{27}\left\{r_1'-\lambda_1\right\}-\frac{20}{27}e^2\left\{r_3'-\lambda_3\right\}+\frac{10}{27}\gamma^2 s_{147}\right\}\sin 4t$$
[131]

$$+\left\{+\frac{38}{17}\left\{r_1'-\lambda_1\right\}-\frac{20}{27}\left\{r_3'-\lambda_3\right\}-\frac{9}{8}\gamma^2 s_{147}\right\}e\sin(4t-x)$$
[132]

$$+\left\{-\frac{20}{27}\left\{r_1'-\lambda_1\right\}-\frac{20}{27}\left\{r_3'-\lambda_3\right\}-\frac{20}{27}\left\{r_4'-\lambda_4\right\}+\frac{3}{8}\gamma^2 s_{147}\right\}e\sin(4t+x)$$
[133]

$$+\left\{-\frac{70}{27}\left\{r_1'-\lambda_1\right\}-\frac{83}{32}e^2\left\{r_3'-\lambda_3\right\}+\frac{21}{16}\gamma^2 s_{147}\right\}e_{\prime}\sin(4t-z)$$
[134]

$$+\left\{+\frac{10}{27}\left\{r_1'-\lambda_1\right\}+\frac{10}{27}\left\{r_3'-\lambda_3\right\}+\frac{3}{16}\gamma^2 s_{147}\right\}e_{\prime}\sin(4t+z)$$
[135]

$$+\left\{-\frac{28}{15}\left\{r_1'-\lambda_1\right\}+\frac{38}{17}\left\{r_3'-\lambda_3\right\}+\frac{15}{16}\gamma^2 s_{147}\right\}e^2\sin(4t-2x)$$
[136]

$$+\left\{-\frac{20}{27}\left\{r_1'-\lambda_1\right\}-\frac{20}{27}\left\{r_4'-\lambda_4\right\}+\frac{3}{8}\gamma^2 s_{147}\right\}e^2\sin(4t+2x)$$
[137]

Development of $\delta\left(\frac{dR}{d\lambda}\right)$.

$$+\left\{+\frac{180}{23}\left\{r_1'-\lambda_1\right\}-\frac{70}{27}\left\{r_3'-\lambda_3\right\}-\frac{63}{16}\gamma^2 s_{147}\right\}e\,e_{,}\sin(4t-x-z)$$

[138]

$$+\left\{+\frac{10}{27}\left\{r_1'-\lambda_1\right\}+\frac{10}{27}\left\{r_4'-\lambda_4\right\}-\frac{3}{16}\gamma^2 s_{147}\right\}e\,e_{,}\sin(4t+x+z)$$

[139]

$$+\left\{-\frac{66}{59}\left\{r_1'-\lambda_1\right\}+\frac{10}{27}\left\{r_3'-\lambda_3\right\}+\frac{9}{16}\gamma^2 s_{147}\right\}e\,e_{,}\sin(4t-x-z)$$

[140]

$$+\left\{-\frac{83}{32}\left\{r_1'-\lambda_1\right\}+\frac{21}{16}\gamma^2 s_{147}\right\}e\,e_{,}\sin(4t+x-z)$$

[141]

$$+\left\{-\frac{233}{37}\left\{r_1'-\lambda_1\right\}+\frac{51}{16}\gamma^2 s_{147}\right\}e_{,}^2\sin(4t-2z)$$

[142]

In order to verify the developments which have been given, suppose

$$R=\frac{38\,m_{,}\,a^2}{17\,a_{,}^3}\,e\cos(2t-x)$$

$$\mathrm{r}\,\delta\,\frac{1}{r}=e_{,}\,r_5'\cos z \qquad\qquad \delta\lambda=e_{,}\,\lambda_5\sin z$$

neglecting δs,

$$\delta R=\left(\frac{dR}{dr}\right)\delta r+\left(\frac{dR}{d\lambda}\right)\delta\lambda=-a\left(\frac{dR}{da}\right)\mathrm{r}\,\delta\,\frac{1}{r}+\left(\frac{dR}{dt}\right)\delta\lambda$$

t being used in the sense $n\,t-n_{,}\,t$.

$$\delta R=-\frac{2\,.\,38\,m_{,}\,a^2}{17\,a_{,}^3}\,e\,e_{,}\left\{\cos(2t-x)\,r_5'\cos z+\sin(2t-x)\,\lambda_5\sin z\right\}$$

$$=\frac{m_{,}\,a^2}{a_{,}^3}\,e\,e_{,}\left\{-\frac{38}{17}r_5'\cos(2t-x+z)-\frac{38}{17}r_5'\cos(2t-x-z)\right.$$

$$\left.+\frac{38}{17}\lambda_5\cos(2t-x+z)-\frac{38}{17}\lambda_5\cos(2t-x-z)\right\}$$

$$=\frac{m_{,}\,a^2}{a_{,}^3}\left\{\underbrace{-\frac{38}{17}\left\{r_5'-\lambda_5\right\}e\,e_{,}\cos(2t-x+z)}_{[15]}\underbrace{-\frac{38}{17}\left\{r_5'+\lambda_5\right\}e\,e_{,}\cos(2t-x-z)}_{[12]}\right\}$$

which terms are in fact given in the development of δR, p. 11 and p. 10.

Again

$$\delta\,\mathrm{d}\,R = \frac{2\,.\,38\,m\,m_{,}\,a^2\,e\,e_{,}}{17\,a_{,}^3}\left\{\cos(2t-x)\,r_5'\sin z - \sin(2t-x)\;\lambda_5\cos z\right\}$$

$$= \frac{m\,m_{,}\,a^2\,e\,e_{,}}{a_{,}^3}\left\{+\frac{38}{17}\,r_5'\sin(2t-x+z) - \frac{38}{17}\,r_5'\sin(2t-x-z)\right.$$

$$\left. - \frac{38}{17}\,\lambda_5\sin(2t-x+z) - \frac{38}{17}\,\lambda_5\sin(2t-x-z)\right\}$$

$$= \frac{m\,m_{,}\,a^2}{a_{,}^3}\left\{+\frac{38}{17}\underset{[15]}{\left\{r_5'-\lambda_5\right\}e\,e_{,}\sin(2t-x+z)} + \frac{38}{17}\underset{[12]}{\left\{r_5'+\lambda_5\right\}e\,e_{,}\sin(2t-x-z)}\right]$$

which terms are given in the development of $\delta\,\mathrm{d}\,R$, p. 18.

Similarly

$$\delta\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\right) = -\frac{2\,.\,2\,.\,38\,m_{,}\,a^2}{17\,a_{,}^3}\,e\,e_{,}\left\{-\sin(2t-x)\,r_5'\cos z + \cos(2t-x)\,\lambda_5\sin z\right\}$$

$$= -\frac{2\,.\,38\,m_{,}\,a^2}{17\,a_{,}^3}\,e\,e_{,}\left\{-r_5'\sin(2t-x+z) - r_5'\sin(2t-x-x)\right.$$

$$\left. + \lambda_5\sin(2t-x+z) - \lambda_5\sin(2t-x-z)\right\}$$

$$= \frac{2\,m_{,}\,a^2}{a_{,}^3}\left\{+\frac{38}{17}\underset{[15]}{\left\{r_5'-\lambda_5\right\}e\,e_{,}\sin(2t-x+z)} + \frac{38}{17}\underset{[12]}{\left\{r_5'+\lambda_5\right\}e\,e_{,}\sin(2t-x-z)}\right]$$

these terms are given in the development of $\delta\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\right)$, p. 25 and p. 24.

Suppose $\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right) = \frac{20\,m_{,}\,a^2}{27\,a_{,}^3}\,\gamma\sin(2t+y) \qquad \delta s = \gamma\,s_{147}\sin(2t-y)$

$$\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right)\delta s = \frac{20\,m_{,}\,a^2}{27\,a_{,}^3}\,\gamma^2 s_{147}\sin(2t+y)\sin(2t-y)$$

$$= \frac{m_{,}\,a^2}{a_{,}^3}\left\{-\underset{[131]}{\frac{10}{27}\,s_{147}\,\gamma^2\cos 4t} + \underset{[62]}{\frac{10}{27}\,s_{147}\,\gamma^2\cos 2y}\right\}$$

which terms are found in the development of $\delta\,R$.

$$\mathrm{d}\,.\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,s}\right)\delta s = -\frac{2\,.\,20\,m\,m_{,}\,a^2}{27\,a_{,}^3}\,\gamma^2 s_{147}\sin(2t+y)\cos(2t-y)$$

$$= \frac{m\,m_{,}\,a^2}{a_{,}^3}\left\{-\underset{[131]}{\frac{20}{27}\,s_{147}\,\gamma^2\sin 4t} + \underset{[62]}{\frac{20}{27}\,s_{147}\,\gamma^2\sin 2y}\right\}$$

which terms are found in the development of $\delta\,\mathrm{d}\,R$. These terms are in fact multiplied by *n* which is equal to *m* if *n* be taken equal to unity.

$$\frac{\mathrm{d}\,.\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\right)}{\mathrm{d}\,s}\,\delta\,s = \frac{2\,.\,20\,m_{,}\,a^{2}}{27\,a_{,}^{3}}\,\gamma^{2}\,s_{147}\cos\left(2\,t+y\right)\sin\left(2\,t-y\right)$$

$$= \frac{2\,m\,a_{,}^{2}}{a_{,}^{3}}\left\{+\underset{[131]}{\frac{10}{27}\,s_{147}\,\gamma^{2}\sin 4\,t} - \underset{[62]}{\frac{10}{27}\,s_{147}\,\gamma^{2}\sin 2\,y}\right\}$$

which terms are found in the development of $\delta\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda}\right)$.

$$\frac{1}{4}\left\{\int\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\lambda}\right)\mathrm{d}\,t\right\}^{2} = \frac{m_{,}^{2}\,a^{4}}{a_{,}^{6}}\left\{\frac{9}{128\,(1-m)^{2}} + \frac{81\,e^{2}}{32\,(2-2\,m-c)^{2}} + \frac{9\,e^{2}}{32\,(2-2\,m+c)^{2}}\right.$$

$$+ \frac{441\,e_{,}^{2}}{128\,(2-3\,m)^{2}} + \frac{9\,e^{2}}{128\,(2-m)^{2}} + \frac{9}{16\,(1-m)^{2}}\,\frac{a^{2}}{a_{,}^{2}} + \underset{[1]}{\frac{9}{16\,(1-m)^{2}}\,\frac{a^{2}}{a_{,}^{2}}\cos 2\,t}$$

$$- \underset{[2]}{\left\{\frac{9}{4\,(2-2\,m-c)} - \frac{3}{4\,(2-2\,m+c)}\right\}\frac{3}{8\,(1-m)}\,e\cos x}$$

$$- \underset{[5]}{\left\{\frac{3}{8\,(2-m)} - \frac{21}{8\,(2-3\,m)}\right\}\frac{3}{8\,(1-m)}\,e_{,}\cos z}$$

$$+ \left\{-\frac{27}{16\,(2-2\,m-c)\,(2-2\,m+c)} + \frac{45}{64\,(1-m)\,(2-2\,m-2\,c)}\right.$$

$$\left. - \frac{9}{32\,(1-m)\,(2-2\,m+2\,c)}\right\}\underset{[8]}{e^{2}\cos 2\,x}$$

$$+ \left[-\left\{\frac{3}{8\,(2-m+c)} + \frac{63}{8\,(2-3\,m-c)}\right\}\frac{3}{8\,(1-m)} + \frac{63}{32\,(2-2\,m+c)\,(2-3\,m)}\right.$$

$$\left. + \frac{9}{32\,(2-2\,m-c)\,(2-m)}\right]\underset{[11]}{e\,e_{,}\cos\left(x+z\right)}$$

$$+ \left[-\left\{\frac{21}{8\,(2-3\,m+c)} + \frac{9}{8\,(2-m-c)}\right\}\frac{3}{8\,(1-m)} - \frac{189}{32\,(2-3\,m)\,(2-2\,m-c)}\right.$$

$$\left. - \frac{9}{32\,(2-m)\,(2-2\,m+c)}\right]\underset{[14]}{e\,e_{,}\cos\left(x-z\right)}$$

$$+ \left\{-\frac{63}{64\,(2-m)\,(2-3\,m)} + \frac{153}{64\,(1-m)\,(2-4\,m)}\right\}\underset{[17]}{e_{,}^{2}\cos 2\,z}$$

$$+\left\{\frac{9}{128(1-m)^2}-\frac{27e^2}{16(2-2m+c)(2-2m-c)}-\frac{63e_{,}^2}{64(2-m)(2-3m)}\right\}\cos 4t$$

[131]

$$-\frac{27}{32(1-m)(2-2m-c)}e\cos(4t-x)+\frac{9}{32(1-m)(2-2m+c)}e\cos(4t+x)$$

[132] [133]

$$+\frac{63}{64(1-m)(2-3m)}e_{,}\cos(4t-z)-\frac{9}{64(1-m)(2-m)}e_{,}\cos(4t+z)$$

[134] [135]

$$+\left\{+\frac{45}{64(1-m)(2-2m-2c)}+\frac{81}{32(2-2m-c)^2}\right\}e^2\cos(4t-2x)$$

[136]

$$+\left\{+\frac{9}{32(1-m)(2-2m+2c)}+\frac{9}{32(2-2m+c)^2}\right\}e^2\cos(4t+2x)$$

[137]

$$+\left\{-\frac{189}{64(1-m)(2-3m-c)}-\frac{189}{32(2-2m+c)(2-3m)}\right\}ee_{,}\cos(4t-x-z)$$

[138]

$$+\left\{-\frac{9}{64(1-m)(2-m+c)}-\frac{9}{32(2-2m+c)(2-m)}\right\}ee_{,}\cos(4t+x+z)$$

[139]

$$+\left\{-\frac{27}{64(1-m)(2-m-c)}+\frac{27}{32(2-2m-c)(2-m)}\right\}ee_{,}\cos(4t-x+z)$$

[140]

$$+\left\{-\frac{63}{64(1-m)(2-m+c)}+\frac{63}{32(2-2m+c)(2-3m)}\right\}ee_{,}\cos(4t+x-z)$$

[141]

$$+\left\{+\frac{153}{64(1-m)(2-4m)}+\frac{441}{128(2-3m)^2}\right\}e_{,}^2\cos(4t-2z)$$

[142]

$$+\frac{9}{128(2-m)^2}e_{,}^2\cos(4t+2z)$$

[143]

$$\frac{h(1+s^2)}{r^2}=\frac{h}{\mathrm{r}^2}+\frac{2h}{\mathrm{r}}\delta.\frac{1}{r}+h\left(\delta\frac{1}{r}\right)^2+\frac{2hs}{\mathrm{r}^2}\delta s+\frac{4hs}{\mathrm{r}}\delta.\frac{1}{r}\delta s+\frac{h\delta s^2}{\mathrm{r}^2}$$

Developments which are required when the cube of the disturbing force is considered.

Neglecting in R the terms multiplied by $\frac{a^3}{a^4}$ and by s^2, and omitting the factor $m_{,}$,

$$R=-\frac{r^2}{4r_{,}^3}\left\{1+3\cos(2\lambda-2\lambda_{,})\right\}$$

$$R = -\frac{(r+\delta r)^2}{4 r_{,}^3}\left\{1 + 3\cos(2\lambda - 2\lambda_{,} + 2\delta\lambda)\right\}$$

$$= -\frac{\{1 + 3\cos(2\lambda - 2\lambda_{,})\}}{4 r_{,}^3}\left\{2 r \delta r + \delta r^2\right\} + \frac{3 r}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\,\delta r\,\delta\lambda$$

$$+ \frac{3}{2}\frac{r^2}{r_{,}^3}(\cos(2\lambda - 2\lambda_{,})(\delta\lambda)^2$$

$$\delta r = -r^2\delta\frac{1}{r} + r^3\left(\delta\frac{1}{r}\right)^2$$

Neglecting the terms multiplied by $\delta\frac{1}{r}$ and $\delta\lambda$,

$$R = -3r^2\frac{\{1 + 3\cos(2\lambda - 2\lambda_{,})\}}{4 r_{,}^3}\left(r\,\delta\frac{1}{r}\right)^2 - \frac{3 r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\left(r\,\delta\frac{1}{r}\right)\delta\lambda_{,}$$

$$+ \frac{3}{2}\frac{r^2}{r_{,}^3}\cos(2\lambda - 2\lambda_{,})(\delta\lambda)^2$$

d R and $r\left(\frac{dR}{dr}\right)$ may be obtained from R as before.

$$\frac{dR}{d\lambda} = \frac{3}{2}\frac{r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})$$

$$\frac{dR}{d\lambda} = \frac{3\{2 r \delta r + \delta r^2\}}{r_{,}^3}\sin(2\lambda - 2\lambda_{,}) + \frac{6\cos(2\lambda - 2\lambda_{,})}{r_{,}^3}\,r\,\delta r\,\delta\lambda - \frac{3 r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})(\delta\lambda)^2$$

Neglecting as before the terms multiplied by $\delta\frac{1}{r}$ and $\delta\lambda$,

$$= \frac{9}{2}\frac{r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})\left(r\,\delta\frac{1}{r}\right)^2 - \frac{6 r^2\cos(2\lambda - 2\lambda_{,})}{r_{,}^3}\left(r\,\delta\frac{1}{r}\right)\delta\lambda$$

$$- \frac{3 r^2}{r_{,}^3}\sin(2\lambda - 2\lambda_{,})(\delta\lambda)^2$$

Retaining the terms depending on the cube of the disturbing force,

$$\frac{d^2 r^2}{2\,dt^2} - \frac{d^2 r^3\,\delta\frac{1}{2}}{dt^2} + \frac{3\,d^2 r^4\left(\delta.\frac{1}{r}\right)^2}{2\,dt^2} - \frac{2\,d^2 r^5\left(\delta\frac{1}{r}\right)^3}{dt^2} - \delta\frac{\mu}{\mu} + 2\int dR + r\left(\frac{dR}{dr}\right) = 0$$

$$\frac{d\lambda'}{dt} = \frac{h}{r^2}\left\{1 - \frac{1}{h}\int\frac{dR}{d\lambda'}dt + \frac{1}{2h^2}\left\{\int\frac{dR}{d\lambda'}dt\right\}^2 - \frac{1.1}{2.4\,h^3}\left\{\int\frac{dR}{d\lambda'}dt\right\}^4\right.$$

$$\left. - \frac{1.1.3}{2.4.6\,h^5}\left\{\int\frac{dR}{d\lambda'}dt\right\}^6 + \&c.\right.$$

Fortunately this series does not appear to contain the quantity $\left\{\frac{dR}{d\lambda}dt\right\}^3$

The principal arguments in the expression for the longitude are those of which the indices and numerical coefficients in seconds (according to M. Damoiseau), ranged in their order of magnitude, are as follows:

$$\lambda' = \underset{[2]}{22639''\cdot70 \sin x} + \underset{[3]}{4589''\cdot61 \sin (2t - x)} + \underset{[1]}{2370''\cdot00 \sin 2t} + \underset{[8]}{768''\cdot72 \sin 2x} - \underset{[5]}{673''\cdot70 \sin z}$$

$$- \underset{[62]}{411''\cdot67 \sin 2y} + \underset{[9]}{211''\cdot57 \sin (2t - 2x)} + \underset{[12]}{207''\cdot09 \sin (2t - x - z)} + \underset{[4]}{192''\cdot22 \sin (2t + x)}$$

$$+ \underset{[6]}{165''\cdot56 \sin (2t - z)} + \underset{[14]}{147''\cdot74 \sin (x - z)} - \underset{[101]}{122''\cdot48 \sin t} - \underset{[11]}{109''\cdot27 \sin (x + z)}$$

The values of the quantities λ are, according to M. Damoiseau, p. 561,

*30.. $\lambda_1 = + \cdot0114901$
31.. $\lambda_3 = + \cdot405714$
32.. $\lambda_4 = + \cdot016992$
16.. $\lambda_5 = - \cdot194501$
33.. $\lambda_6 = + \cdot047798$
34.. $\lambda_7 = - \cdot0071657$
35.. $\lambda_9 = + \cdot341010$
36.. $\lambda_{10} = + \cdot023758$
19.. $\lambda_{11} = - \cdot57521$
41.. $\lambda_{12} = + 1\cdot090142$
42.. $\lambda_{13} = - \cdot015950$
18.. $\lambda_{14} = + \cdot77772$
39.. $\lambda_{15} = - \cdot15092$
40.. $\lambda_{16} = + \cdot077330$
17.. $\lambda_{17} = - \cdot12619$
43.. $\lambda_{18} = + \cdot13616$
44.. $\lambda_{19} = - \cdot0056734$
45.. $\lambda_{21} = + \cdot37647$
46.. $\lambda_{22} = + \cdot037323$
21.. $\lambda_{23} = - \cdot73617$
53.. $\lambda_{24} = + \cdot86288$
54.. $\lambda_{25} = - \cdot018236$
20.. $\lambda_{26} = + \cdot93487$
51.. $\lambda_{27} = + \cdot24475$
52.. $\lambda_{28} = + \cdot086383$
23.. $\lambda_{29} = - \cdot35736$
59.. $\lambda_{30} = + 2\cdot35733$
22.. $\lambda_{32} = + \cdot78995$
61.. $\lambda_{33} = - \cdot81190$

60.. $\lambda_{34} = + \cdot1630$
67.. $\lambda_{39} = + \cdot49834$
27.. $\lambda_{41} = - \cdot68253$
73.. $\lambda_{42} = + \cdot84004$
26.. $\lambda_{44} = + \cdot99754$
75.. $\lambda_{48} = + \cdot62872$
37.. $\lambda_{63} = + \cdot032768$
38.. $\lambda_{64} = - \cdot0034364$
49.. $\lambda_{67} = + \cdot0029421$
47.. $\lambda_{68} = - \cdot105155$
48.. $\lambda_{69} = - \cdot072464$
50.. $\lambda_{70} = - \cdot010897$
24.. $\lambda_{71} = - \cdot001424$
25.. $\lambda_{72} = + \cdot013168$
57.. $\lambda_{73} = + \cdot10392$
56.. $\lambda_{74} = - \cdot0060501$
55.. $\lambda_{75} = - \cdot047688$
58.. $\lambda_{76} = - \cdot0003559$
65.. $\lambda_{79} = - \cdot10332$
63.. $\lambda_{80} = - \cdot11921$
64.. $\lambda_{81} = - \cdot10332$
80.. $\lambda_{101} = - \cdot235981 \dfrac{a}{a_{\prime}}$
81.. $\lambda_{102} = - \cdot60389 \dfrac{a}{a_{\prime}}$
82.. $\lambda_{103} = - \cdot29509 \dfrac{a}{a_{\prime}}$
83.. $\lambda_{104} = - \cdot055072 \dfrac{a}{a_{\prime}}$

84.. $\lambda_{105} = + 2\cdot0147 \dfrac{a}{a_{\prime}}$
85.. $\lambda_{106} = - \cdot74945 \dfrac{a}{a_{\prime}}$
86 . $\lambda_{107} = - \cdot30746 \dfrac{a}{a_{\prime}}$
91.. $\lambda_{108} = - \cdot60668 \dfrac{a}{a_{\prime}}$
92.. $\lambda_{109} = + \cdot26150 \dfrac{a}{a_{\prime}}$
89.. $\lambda_{110} = + 4\cdot29 \dfrac{a}{a_{\prime}}$
100.. $\lambda_{116} = + \cdot00001927 \dfrac{a}{a_{\prime}}$
101.. $\lambda_{117} = - \cdot10679 \dfrac{a}{a_{\prime}}$
102.. $\lambda_{118} = + \cdot011593 \dfrac{a}{a_{\prime}}$
103.. $\lambda_{119} = + \cdot010326 \dfrac{a}{a_{\prime}}$
104.. $\lambda_{120} = + \cdot009179 \dfrac{a}{a_{\prime}}$
120.. $\lambda_{131} = + \cdot000071995$
121.. $\lambda_{132} = + \cdot0034139$
122.. $\lambda_{133} = - \cdot0000380$
123.. $\lambda_{134} = + \cdot0002367$
124.. $\lambda_{135} = - \cdot00002598$
125.. $\lambda_{136} = + \cdot050272$
126.. $\lambda_{137} = + \cdot0011766$
131.. $\lambda_{138} = + \cdot017477$
129.. $\lambda_{140} = - \cdot0025794$
127.. $\lambda_{144} = + \cdot00037053$

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* Indices of M. Damoiseau.

According to the value of the parallax given by M. DAMOISEAU, p. 573,

$r_1 = \cdot 00834,\ r_3 = \cdot 18350,\ r_4 = \cdot 01625,\ r_5 = \cdot 00547,\ r_6 = \cdot 03342,\ r_7 = \cdot 004525,$ &c. nearly.

From the preceding values it appears that several of the quantities λ which correspond to arguments in the longitude depending on the cubes and fourth powers of the eccentricities are of the same order as those which correspond to the arguments 1, 3, &c.: hence in order to carry the development of δR and $\delta \frac{dR}{d\lambda}$, &c. to the terms depending on the cubes of the eccentricities, λ_{21}, λ_{23}, λ_{24}, &c. cannot be neglected when extreme accuracy is sought; and if the method which I have employed should be adopted, it will be necessary to extend very considerably the Table II. so as to embrace these quantities.

The advantages of this method appear to me by no means confined to the condition of taking into account all sensible quantities; a few lines of calculation suffice to obtain approximate results.

Thus neglecting the squares of the eccentricities,

$$R = m_{\prime} \left\{ -\frac{1}{r_{\prime}} - \frac{a^2}{4 a_{\prime}^3} - \frac{3}{4}\frac{a^2}{a_{\prime}^3}\cos 2t + \frac{a^2}{2a_{\prime}^3} e \cos x + \frac{9}{4}\frac{a^2}{a_{\prime}^3} e \cos (2t - x) - \frac{3}{4}\frac{a^2}{a_{\prime}^3} e \cos (2t + x) \right.$$
$$\left. - \frac{3}{4}\frac{a^2}{a_{\prime}^3} e_{\prime} \cos z - \frac{21}{8}\frac{a^2}{a_{\prime}^3} e_{\prime} \cos (2t - z) + \frac{3}{8}\frac{a^2}{a_{\prime}^3} e_{\prime} \cos (2t + z) \right\}$$

$$- r_0 - \frac{m_{\prime} a^3}{2 m a_{\prime}^3} = 0$$

$$4(1-m)^2 r_1 - r_1 - \frac{3}{2}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{1}{1-m} + 1\right\} = 0$$

$$c^2 \left\{1 - 3 r_0\right\} - 1 + \frac{2 m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3} = 0$$

$$(1-2m)^2 \left\{r_3 - \frac{3}{2} r_1\right\} - r_3 + \frac{9}{2}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{1}{1-2m} + 1\right\} = 0$$

$$(3-2m)^2 \left\{r_4 - \frac{3}{2} r_1\right\} - r_4 - \frac{3}{2}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{3}{3-2m} + 1\right\} = 0$$

$$m^2 r_5 - r_5 - \frac{3}{2}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3} = 0$$

$$(2-3m)^2 r_6 - r_6 - \frac{21}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{2}{2-3m} + 1\right\} = 0$$

$$(2-m)^2 r_7 - r_7 + \frac{3}{4}\frac{m_{\prime}}{\mu}\frac{a^3}{a_{\prime}^3}\left\{\frac{2}{2-m} + 1\right\} = 0$$

$$-\left\{1-2m\right\}^2 z_{147}+\frac{3r_1}{2}+z_{147}=0$$

$$\lambda=\frac{h}{a^2}\left\{1+2r_0\right\}t+\frac{2e(1+r_0)}{c}\sin x$$

$$+\left\{2r_1+\frac{3m_{,}a^3}{4(1-m)\mu a_{,}^3}\right\}\frac{1}{2(1-m)}\sin 2t$$

$$+\left\{2r_3+r_1-\left\{\frac{9}{2(1-m)}-\frac{3}{2(2-m)}\right\}\frac{m_{,}a^3}{\mu a_{,}^3}\right\}\frac{e}{(1-2m)}\sin(2t-x)$$

$$+\left\{2r_4+r_1-\left\{-\frac{3}{2(3-m)}-\frac{3}{2(2-m)}\right\}\frac{m_{,}a^3}{\mu a_{,}^3}\right\}\frac{e}{(3-m)}\sin(2t+x)$$

$$+\frac{2r_5e_{,}}{m}\sin z$$

$$+\left\{2r_6+\frac{21m_{,}a^3}{4(2-3m)\mu a_{,}^3}\right\}\frac{e_{,}}{(2-3m)}\sin(2t-z)$$

$$+\left\{2r_7-\frac{3m_{,}a^3}{4(2-m)\mu a_{,}^3}\right\}\frac{e_{,}}{(2-m)}\sin(2t+z)$$

The values of the constants assumed by M. DAMOISEAU are

$e=\cdot 0548442$ $e_{,}=\cdot 0167927$ $\gamma=\cdot 0900684$ $m=\cdot 0748013$

Mém. Théor. Lun. p. 502.

Taking $m=\frac{3}{40}=\cdot 075$

$$\frac{4\,.\,37^2}{40^2}r_1-r_1-\frac{3\,.\,77\,m_{,}a^3}{2\,.\,37\,\mu a_{,}^3}=0 \qquad r_1=\frac{600\,.\,77\,m_{,}a^3}{17\,.\,57\,.\,37\,\mu a_{,}^3}$$

$$\frac{34^2}{40^2}\left\{r_3-\frac{3}{2}r_1\right\}-r_3+\frac{9\,.\,74\,m_{,}a^3}{2\,.\,34\,\mu a_{,}^3}=0$$

$$r_3=300\left\{\frac{17\,.\,77}{37^2\,.\,57}-\frac{2}{17}\right\}\frac{m_{,}a^3}{\mu a_{,}^3}=\frac{300\,.\,133813\,m_{,}a^3}{1326561\,\mu a_{,}^3}$$

$$r_5=-\frac{40^2\,.\,3\,m_{,}a^3}{37\,.\,43\,.\,2\,\mu a_{,}^3}=-\frac{2400\,m_{,}a^3}{1591\,\mu a_{,}^3}$$

$$z_{147}=\frac{3\,.\,40^2\,r_1}{2\,.\,4\,.\,37\,.\,3}$$

$$\lambda_1=\left\{2r_1+\frac{3\,.\,40\,m_{,}a^3}{4\,.\,37\,\mu a_{,}^3}\right\}\frac{20}{37}$$

$$\lambda_3=\left\{2r_3+r_1-\left\{\frac{9\,.\,20}{37}-\frac{3\,.\,20}{77}\right\}\frac{m_{,}a^3}{\mu a_{,}^3}\right\}\frac{20}{17}$$

$$= \left\{ 2 r_3 + r_1 - \frac{11640 m_, a^3}{2849 \mu a_,^3} \right\} \frac{20}{17}$$

$$\lambda_5 = - \frac{40^3 m_, a^3}{37 \, . \, 43 \, \mu a_,^3}$$

If $\frac{m_, a^3}{\mu a_,^3} = \frac{1}{178{\cdot}725}$, as Newton finds, Principia, vol. iv. p. 2, Glasgow edit.,

$$r_1 = {\cdot}007208 \qquad r_3 = {\cdot}16928 \qquad r_5 = - {\cdot}008437 \qquad z_{147} = {\cdot}03896$$

$$\lambda_1 = {\cdot}010244 \qquad \lambda_3 = {\cdot}40409 \qquad \lambda_5 = - {\cdot}22501 \qquad s_{147} = z_{147} + r_1 = {\cdot}04617$$

These values being substituted in the developments of δR, $\delta \, \mathrm{d} R$ &c. given in this paper, more accurate values may be found from the differential equations by a new integration. It would be shorter, but perhaps not quite so satisfactory, to assume the values of λ_1, λ_3, &c. given by M. Damoiseau in these substitutions.

Converting the coefficients of the arguments of longitude into sexagesimal seconds;

$$\lambda = \underset{2370}{2113''} \sin 2t + \underset{4589{\cdot}61}{4571''{\cdot}3} \sin (2t - x) - \underset{673{\cdot}7}{779''{\cdot}3} \sin z$$

The numbers underneath are the values according to M. Damoiseau.

The coefficient of the variation thus obtained (2113″ or 35′ 13″) agrees within three seconds with that found by Newton, vol. iv. p. 19, which is 35′ 10″. The approximation is in fact of the same order as that of Newton. Newton does not appear to have succeeded in determining the evection, the most considerable of all the lunar inequalities after the equation of the centre. The value assigned by him to the annual equation is 11′ 51″ or 711″ (corresponding to $e_, = {\cdot}0169166$); he has not however given the method by which it was obtained.

The equation

$$c^2 \{1 - 3 r_0\} - 1 + \frac{m'}{\mu} \left\{ - \frac{a^3}{2 a_,^3} b_{3,0} + \frac{a^2}{a_,^2} b_{3,1} \right\} = 0$$

since $r_0 = \frac{m_,}{\mu} \left\{ \frac{a^3}{2 a_,^3} b_{3,0} - \frac{a^2}{2 a_,^3} b_{3,1} \right\}$ (See Phil. Trans. 1831. p. 50.)

gives $c = 1 + \frac{m_,}{\mu} \left\{ \frac{a^3}{a_,^3} b_{3,0} - \frac{5 a^2}{4 a_,^2} b_{3,1} \right\}$

If $\frac{h}{a^2} \left\{ 1 + 2 r_0 \right\} = \mathrm{n}$ or $n \left\{ 1 + 2 r_0 \right\} = \mathrm{n}$

$$c n = \mathrm{n} \left\{ 1 - \frac{m_,}{\mu} \left\{ \frac{a^3}{a_,^3} b_{3,0} - \frac{a^2}{a_,^2} b_{3\,1} - \frac{a^3}{a_,^3} b_{3,0} + \frac{5}{4} \frac{a^2}{a_,^2} b_{3,1} \right\} \right\}.$$

$$c\,n = \mathrm{n}\left\{1 - \frac{m_{,}a^2}{4\mu\, a_{,}^2}\, b_{3,1}\right\}$$

$$= \mathrm{n}\left\{1 - \frac{3\,m_{,}}{4\mu}\,\frac{a^3}{a_{,}^3}\right\} \text{ nearly.}$$

This coincides with the first term of the expression, Math. Tracts, p. 59.

If $\sqrt{\frac{\mu}{a^3}}\left\{1 - \frac{m_{,}}{\mu}\,\frac{a^3}{a_{,}^3}\right\} = \sqrt{\frac{\mu}{\mathrm{a}^3}}$

$$a = \mathrm{a}\left\{1 - \frac{2\,m_{,}}{3\mu}\,\frac{a^3}{a_{,}^3}\right\}$$

$$\frac{1 + r_0}{a} = \frac{1 + \frac{2\,m_{,}}{3\mu}\,\frac{a^3}{a_{,}^3} - \frac{m_{,}a^3}{2\,a_{,}^3}}{\mathrm{a}} = \frac{1 + \frac{m_{,}}{6\mu}\,\frac{a^3}{a_{,}^3}}{\mathrm{a}}$$

The equation for determining z gives

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \frac{\mu z}{r^3} + \left(\frac{\mathrm{d}R}{\mathrm{d}z}\right) = 0$$

If $\quad s = \gamma \sin(g\,n\,t + \epsilon - \nu)$

$$-g^2 + 1 + 3\,r_0 + \frac{m_{,}}{2\mu}\,\frac{a^3}{a_{,}^3}\, b_{3,0} = 0$$

$$r_0 = \frac{m_{,}}{\mu}\left\{\frac{a^3}{2\,a_{,}^3}\, b_{3,0} - \frac{a^2}{2\,a_{,}^2}\, b_{3,1}\right\}$$

$$g^2 = 1 + \frac{m_{,}}{\mu}\left\{\frac{3\,a^3}{2\,a_{,}^3}\, b_{3,0} - \frac{3\,a^2}{2\,a_{,}^2}\, b_{3,1} + \frac{a^3}{2\,a_{,}^3}\, b_{3,0}\right\} = 0$$

$$g = 1 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{a_{,}^3}\, b_{3,0} - \frac{3}{4}\,\frac{a^2}{a_{,}^2}\, b_{3,1}\right\}$$

$$n = \mathrm{n}\{1 - 2\,r_0\}$$

$$g\,n = \mathrm{n}\left\{1 + \frac{m_{,}}{\mu}\left\{\frac{a^3}{a_{,}^3}\, b_{3,0} - \frac{3}{4}\,\frac{a^2}{a_{,}^2}\, b_{3,1}\right\}\right\}\left\{1 - \frac{m_{,}}{\mu}\left\{\frac{a^3}{a_{,}^3}\, b_{3,0} - \frac{a^2}{a_{,}^2}\, b_{3,1}\right\}\right\}$$

$$= \mathrm{n}\left\{1 + \frac{m_{,}}{4\mu}\,\frac{a^2}{a_{,}^2}\, b_{3,1}\right\}$$

$$= \mathrm{n}\left\{1 + \frac{3}{4}\,\frac{m_{,}}{\mu}\,\frac{a^3}{a_{,}^3}\right\} \text{ nearly.}$$

This also coincides with the first term of the expression, Math. Tracts, p. 59; and it appears that when the square of the disturbing force is neglected, the *mean motion* of the perihelium of a planet is retrograde and equal to the *mean motion* of its node taken with a contrary sign.

The equations

$$\mathrm{d}\nu + \frac{r'^2 \sin(\lambda' - \nu)}{h^2 \tan\iota}\left\{(1 + s^2)\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right) - r's\left(\frac{\mathrm{d}R}{\mathrm{d}r'}\right) - \left(\frac{\mathrm{d}R}{\mathrm{d}\lambda'}\right)\left(\frac{\mathrm{d}s}{\mathrm{d}\lambda'}\right)\right\}\mathrm{d}\lambda' = 0$$

$$d\iota + \frac{r'^2 \cos\iota^2 \cos(\lambda' - \nu)}{h^2}\left\{(1+s^2)\right\}\left(\frac{dR}{ds}\right) - r's\left(\frac{dR}{dr'}\right) - \left(\frac{dR}{d\lambda'}\right)\left(\frac{ds}{d\lambda'}\right)\Big\}\, d\lambda' = 0$$

(see Phil. Trans. 1830, p. 334), serve to verify some of the theorems of Newton in the third volume of the Principia.

In fact

$$R = -\frac{m_{,} r'^2}{4 r_{,}^3}\left\{1 + 3\cos(2\lambda - 2\lambda_{,}) - 2s^2\right\}$$

$$(1+s^2)\left(\frac{dR}{ds}\right) = m_{,} s (1+s^2)\frac{r'^2}{r_{,}^3}$$

$$r's\left(\frac{dR}{dr'}\right) = -\frac{m_{,} r'^2}{2 r_{,}^3}\left\{1 + 3\cos(2\lambda - 2\lambda_{,}) - 2s^2\right\} s$$

$$\left(\frac{dR}{d\lambda'}\right) = \frac{3 m_{,} r'^2}{2 r_{,}^3}\sin(2\lambda' - 2\lambda_{,}) \qquad \frac{ds}{d\lambda'} = \tan\iota\cos(\lambda' - \nu)$$

neglecting s^3,

$$d\nu + \frac{m_{,} r'^4 \sin(\lambda - \nu)}{h^2 r_{,}^3}\left\{\sin(\lambda - \nu) + \frac{\sin(\lambda-\nu)}{2}\left\{1 + 3\cos(2\lambda - 2\lambda_{,})\right\}\right.$$
$$\left. - \frac{3}{2}\cos(\lambda - \nu)\sin(2\lambda - 2\lambda_{,})\right\} d\lambda = 0$$

$$d\nu + \frac{m_{,} r^4 \sin(\lambda - \nu)}{h^2 r_{,}^3}\left\{\sin(\lambda - \nu) + 3\cos(\lambda - \lambda_{,})\left\{\cos(\lambda - \lambda_{,})\sin(\lambda - \nu)\right.\right.$$
$$\left.\left. - \sin(\lambda - \lambda_{,})\cos(\lambda - \nu)\right\} - \sin(\lambda - \nu)\right\} d\lambda = 0.$$

$$d\nu = \frac{3 m_{,} r^4}{h^2 r_{,}^3}\sin(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda'$$

$$= \frac{3 m_{,} a^3}{\mu a_{,}^3}\sin(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda \text{ nearly}$$

$$= \frac{1}{59{\cdot}575}\sin(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda$$

Which agrees with the result of Newton, Prop. Lib. 3. "Est igitur velocitas nodorum ut IT × PH × AZ, sive ut contentum sub sinubus trium angulorum TPI, PTN et STN. ..... Sunto enim PK, PH et AZ prædicti tres sinus. Nempe PK sinus distantiæ Lunæ a quadraturâ, PH sinus distantiæ Lunæ a nodo et AZ sinus distantiæ nodi a Sole, et erit velocitas nodi ut contentum PK × PH × AZ."

Similarly

$$d\iota = \frac{3 m_{,} r^4 \cos\iota}{h^2 r_{,}^3}\sin\iota\cos(\lambda - \nu)\cos(\lambda - \lambda_{,})\sin(\lambda_{,} - \nu)\, d\lambda$$

" Erit angulus GP*g* (seu inclinationis horariæ variatio) ad angulum 33″ 16‴ 3⁗ ut IT × AZ × TG × $\frac{\text{P}p}{\text{PG}}$ ad AT cub." Prop. XXXIV.

The stability of the system requires that the quantities *c* and *g*, which are determined by quadratic equations, should be rational. This is the case in the Theory of the Moon.

In the Planetary Theory, by well known theorems,

$$\mathrm{d}\,\epsilon = (1 - \sqrt{1-e^2})\,\mathrm{d}\,\varpi + \frac{2\,a^2 n}{\mu}\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,a}\right)\mathrm{d}\,t$$

$$\mathrm{d}\,\varpi = -\,a n\,\frac{\sqrt{1-e^2}}{\mu\,e}\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,e}\right)\mathrm{d}\,t \qquad \mathrm{d}\,\nu = \frac{a n}{\mu \sin\iota\sqrt{1-e^2}}\left(\frac{\mathrm{d}\,R}{\mathrm{d}\,\iota}\right)\mathrm{d}\,t$$

Neglecting the terms which are periodical,

$$\frac{\mathrm{d}\,\epsilon - \mathrm{d}\,\varpi}{\mathrm{d}\,t} = \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}\,b_{3,0} - \frac{5}{4}\,\frac{a^2}{a_{\prime}^2}\,b_{3,1}\right\} = k = -\,{}^{*}\frac{7}{4}\,\frac{m_{\prime}}{\mu}\,\frac{a^3}{a_{\prime}^3}$$

$$\frac{\mathrm{d}\,\epsilon - \mathrm{d}\,\nu}{\mathrm{d}\,t} = \frac{m_{\prime}}{\mu}\left\{\frac{a^3}{a_{\prime}^3}\,b_{3,0} - \frac{3}{4}\,\frac{a^2}{a_{\prime}^2}\,b_{3,1}\right\} = -\,\frac{m_{\prime}}{4\,\mu}\,\frac{a^3}{a_{\prime}^3}$$

which evidently coincides with the result given p. 38.

Considering the parallactic inequality,

$$(1-m)^2\,r_{101} - r_{101} - \frac{3\,m_{\prime}\,a^4}{8\,\mu\,a_{\prime}^4}\left\{\frac{2}{(1-m)} + 3\right\} = 0$$

$$\lambda_{101} = \left\{2\,r_{101} - \frac{3}{8\,(1-m)}\,\frac{m_{\prime}\,a^4}{\mu\,a_{\prime}^4}\right\}\frac{1}{(1-m)}$$

$$r_{101} = -\,\frac{191\,.\,200}{77\,.\,37}\,\frac{m_{\prime}\,a^4}{\mu\,a_{\prime}^4}$$

$$\lambda_{101} = \left\{2\,r_{101} + \frac{3\,.\,5}{37}\,\frac{m_{\prime}\,a^4}{\mu\,a_{\prime}^4}\right\}\frac{40}{37}$$

which equations give $r_{101} = -\,{\cdot}07521\,\frac{a}{a_{\prime}}$; and if the parallactic inequality $= 122''{\cdot}38$ according to Burg, and $a = \frac{1}{57'}$ or $\frac{1}{3420''}$, $a_{\prime} = \frac{1}{12''{\cdot}67}$, that is, if the moon's horizontal parallax $= 57'$, the sun's parallax, according to the preceding equations, is 12″·7 ; which however differs widely from the accurate value 8″·54.

When the square of the disturbing force is neglected, the variable part of the angle $t + z$ may be considered the same as that of the angle x, and there-

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[*] $b_{3,0} = 2\left\{1 + \left(\frac{3}{2}\right)^2\frac{a^2}{a_{\prime}^2} + \text{\&c.}\right\} \qquad b_{3,1} = \frac{3\,a}{a_{\prime}} + \frac{3\,.\,3\,.\,5}{2\,.\,4}\,\frac{a^3}{a_{\prime}^3} + \text{\&c.}$

fore they may be included in the same inequality, either in the expression for the parallax or in that for the mean longitude.

In the elliptic theory

$$\frac{h^2}{\mu \cos \iota^2} = a\,(1 - e'^2)$$

$$e'^2 = e^2\{1 - \sin^2 \iota \sin^2 (v - \varpi)\}$$ See Phil. Trans. 1831, p. 56.

These equations of condition are true, however far the approximation be carried; provided only, that the arbitrary quantities e and $\sin \iota$ be determined so as not to contain the mass of the sun implicitly.

The determination of the coefficients of the arguments $t + z$, $t - x + z$, and $2\,t - 2\,x + 2\,z$ will require particular attention in the numerical calculation. According to the analysis of M. Poisson (Journal de l'Ecole Polytechnique, vol. viii. and Mémoires de l'Académie des Sciences, vol. i.), the coefficient of the argument $t - x + z$ in the quantity $\int \mathrm{d}\, R$ equals zero. Conversely therefore this theorem may furnish an equation of condition between some of the coefficients. According to M. Damoiseau, the coefficient of this argument in the expression for the longitude is only $2''\cdot 05$, and the argument $2\,t - 2\,x + 2\,z$ is insensible. The expressions which I gave, Phil. Trans. 1830, p. 334, are well adapted for finding in the theory of the moon, in which the square of the disturbing force is so sensible, by means of the variation of the elliptic constants, the coefficient of any inequality which arises from the introduction of a small divisor, these expressions being true, however far the approximation is carried.

It may be seen in the authors themselves, or in the excellent history of physical astronomy by M. Gautier *, that the methods of Clairaut, D'Alembert and Euler, do not resemble in any respect those which I have employed. Both Clairaut and D'Alembert, by means of the differential equation of the second order in which the true longitude is the independent variable, obtained the expression for the reciprocal of the radius vector in terms of cosines of the true longitude. They substituted this value in the differential equation which determines the time, and obtained by integration the value of the mean motion in terms of sines of the true longitude. By the reversion of series they then found the true longitude in terms of sines of the mean motion. The method

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* Essai Historique sur le Problème des Trois Corps, p. 53.

of Euler is not so simple, but is remarkable as introducing the employment of three rectangular coordinates and the decomposition of forces in the direction of three rectangular axes.

Although D'Alembert and Clairaut made use of the same differential equations, disguised under a different notation *, yet they did not arrive at these in the same manner, nor did they employ the same method of integration.

Laplace has pushed the approximations to a much greater extent; but his method coincides in all respects with that of Clairaut.

In the method of Clairaut, when the square of the disturbing force and the squares of the eccentricity and inclination are neglected, the equations employed are

$$\left\{\frac{d^2\frac{1}{r}}{d\lambda^2}+\frac{1}{r}\right\}\left\{1-\frac{2}{h^2}\int r^2\left(\frac{dR}{d\lambda}\right)d\lambda\right\}-\frac{1}{a}$$

$$-\frac{r}{h^2}\left\{r\left(\frac{dR}{dr}\right)-\frac{1}{r}\left(\frac{dR}{d\lambda}\right)\frac{dr}{d\lambda}\right\}=0$$

$$\frac{dR}{d\lambda}=\frac{3m_{,}r^2}{2r_{,}^3}\sin(2\lambda-2\lambda_{,})$$

$$\frac{dR}{dr}=-\frac{m_{,}r}{2r_{,}^3}\left\{1+3\cos(2\lambda-2\lambda_{,})\right\}$$

$$h^2=\mu a \qquad r=\frac{a}{1+e\cos(c\lambda-\varpi)} \qquad \frac{dr}{d\lambda}=ce\sin(c\lambda-\varpi)$$

$$\frac{d^2\frac{1}{r}}{d\lambda^2}+\frac{1}{r}-\frac{1}{a}-\frac{3m_{,}}{\mu a^2}\int\frac{r^4}{r_{,}^3}\sin(2\lambda-2\lambda_{,})\,d\lambda$$

$$+\frac{m_{,}}{2\mu}\frac{r^3}{a r_{,}^3}\left\{1+3\cos(2\lambda-2\lambda_{,})\right\}$$

$$+\frac{3m_{,}er^2}{2\mu r_{,}^3}\sin(2\lambda-2\lambda_{,})\sin(\lambda-\varpi)=0$$

In order to integrate this equation, the value of $\lambda_{,}$ in terms of λ must be substituted, which substitution is an operation by no means simple, and therefore liable to occasion error.

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* The neglect by mathematicians of care in the selection of algebraical symbols is much to be regretted. "La clarté des idées augmente à mesure que l'on perfectionne les signes qui servent à les exprimer."

$$\lambda_{\prime} = m\lambda - 2me\sin(c\lambda - \varpi) + 2e_{\prime}\sin(cm\lambda - \varpi_{\prime}) + \&c.$$

The equation

$$h\,dt = d\lambda\left\{1 + \frac{1}{h^2}\int\left(\frac{dR}{d\lambda}\right)r^2\,d\lambda\right\}$$

gives t in terms of λ, and by the reversion of series λ may afterwards be obtained in terms of t. The equation for determining the inequalities of latitude is

$$\left\{\frac{d^2 s}{d\lambda^2} + s\right\}\left\{1 - \frac{2}{h^2}\int\left(\frac{dR}{d\lambda}\right)r^2\,d\lambda\right\}$$

$$+ \frac{r^2}{h^2}\left(\frac{dR}{ds}\right) - \frac{r^2 s}{h^2}\left(\frac{dR}{dr}\right) - \frac{r^2}{h^2}\left(\frac{dR}{d\lambda}\right)\left(\frac{d\lambda}{ds}\right) = 0$$

$$\frac{dR}{ds} = \frac{3m_{\prime}}{2}\frac{r^2}{r_{\prime}^3}\left\{1 + \cos(2\lambda - 2\lambda_{\prime})\right\} \qquad \frac{ds}{d\lambda} = g\gamma\cos(g\lambda - \nu)$$

I have given these equations, (which are to be found in various works *,) for the convenience of reference.

On the Planetary Theory.

In a former paper I have shown how the coefficients of the terms in the disturbing function multiplied by the cubes of the eccentricities in some particular examples may be reduced by means of some transformations applied to the coefficients of the same function multiplied by the squares of the eccentricities. The form of the disturbing function thus obtained is I think simpler than that of the Méc. Cél. in the terms multiplied by the cubes of the eccentricities, although the advantage obtained by these reductions is not so great as in the case of the terms multiplied by the squares of the eccentricities. I have now given the *general* form of the transformations required, in case any one should think it worth while to extend to the cubes of the eccentricities the general expression for the disturbing function given in the Philosophical Transactions for 1831, p. 295.

The coefficient of $e e_{\prime}\cos(2nt - 4n_{\prime}t + \varpi + \varpi_{\prime})$ or $e e_{\prime}\cos(3t - x + z)$

$$= \frac{3a}{4.4a_{\prime}^2}b_{3,2} + \frac{3a}{2.4a_{\prime}^2}b_{3,4} + \frac{3.3a^2}{2.4.2a_{\prime}^3}b_{5,1} + \frac{3}{2.4}\frac{(3a^2 - a_{\prime}^2)aa_{\prime}}{a_{\prime}^5}b_{5,2} - \frac{3.7a^2}{2.4a_{\prime}^3}b_{5,2}$$

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* See The Mechanism of the Heavens, by Mrs. SOMERVILLE, p. 427.

$$+\frac{3}{2.4}\frac{(3a^2-a_{,}^2)}{a_{,}^5}b_{5,4}+\frac{3.3\,a^2}{2.4.2\,a_{,}^3}b_{5,5}$$

Changing b_3 into $-\frac{3}{4}b_5$, and b_5 into $-\frac{5}{6}b_7$, we have

$$-\frac{3.3}{2.4.4}\frac{a}{a_{,}^2}b_{5,2}-\frac{3.3}{4.2.4}\frac{a}{a_{,}^2}b_{5,4}-\frac{5.3.3}{6.2.4.2}\frac{a^2}{a_{,}^3}b_{7,1}-\frac{5.3}{6.2.4}\frac{(3a_{,}^2-a^2)\,a\,a_{,}}{a_{,}^5}b_{7,2}$$

$$+\frac{5.3.7}{6.2.4}\frac{a^2}{a_{,}^3}b_{7,3}-\frac{5.3}{6.2.4}\frac{(3a^2-a_{,}^2)\,a\,a_{,}}{a_{,}^5}b_{7,4}-\frac{5.3.3}{6.2.4.2}\frac{a^2}{a_{,}^3}b_{7,5}$$

$$=-\frac{3.3}{2.4.4}\frac{a}{a_{,}^2}b_{5,2}-\frac{3.3a}{4.2.4a_{,}^2}b_{5,4}-\frac{5.3.3}{6.2.4.2}\frac{a^2}{a_{,}^3}b_{7,1}-\frac{5.3.3}{6.2.4}\frac{(a^2+a_{,}^2)\,a\,a_{,}}{a_{,}^2}b_{7,2}$$

$$+\frac{5.3}{6.2}\frac{a^2}{a_{,}^3}b_{7,2}+\frac{5.3.7}{6.2.4}\frac{a^2}{a_{,}^3}b_{7,3}+\frac{5.3}{6.2.4}\frac{(a^2+a_{,}^2)\,a\,a_{,}}{a_{,}^5}b_{7,4}$$

$$-\frac{5.3}{6.2}\frac{a}{a_{,}^4}b_{7,4}-\frac{5.3.3}{6.2.4.2}\frac{a^2}{a_{,}^3}b_{7,5}$$

$$=-\frac{3.3}{2.4.4}\frac{a}{a_{,}^2}b_{5,2}-\frac{3.3}{4.2.4}\frac{a}{a_{,}^2}b_{5,4}-\frac{5.3}{4.4}\frac{a}{a_{,}^2}\left\{\frac{(a^2+a_{,}^2)}{a_{,}^2}b_{7,2}-\frac{a}{a_{,}}b_{7,1}-\frac{a}{a_{,}}b_{7,3}\right\}$$

$$+\frac{5.3}{6.2}\frac{a^3}{a_{,}^4}b_{7,2}+\frac{5.3}{6.2.4}\frac{a}{a_{,}^2}\left\{\frac{(a^2+a_{,}^2)}{a_{,}^2}b_{7,4}-\frac{a}{a_{,}}b_{7,3}-\frac{a}{a_{,}}b_{7,5}\right\}$$

$$-\frac{5.3.3}{2.4.4}\frac{a^2}{a_{,}^3}\left\{b_{7,1}-b_{7,3}\right\}+\frac{5}{2.4.4}\frac{a^2}{a_{,}^3}\left\{b_{7,3}-b_{7,5}\right\}-\frac{5.3}{6.2}\frac{a^3}{a_{,}^4}b_{7,4}$$

$$=-\frac{3.3}{2.4.4}\frac{a}{a_{,}^2}b_{5,2}-\frac{3.3}{4.2.4}\frac{a}{a_{,}^2}b_{5,4}-\frac{5.3}{4.4}\frac{a}{a_{,}^2}b_{5,2}+\frac{3.3}{6}\frac{a}{a_{,}^2}b_{5,3}$$

$$+\frac{5.3}{6.2.4}\frac{a}{a_{,}^2}b_{5,4}-\frac{2.3.3}{4.4}\frac{a}{a_{,}^2}b_{5,2}+\frac{4}{4.4}\frac{a}{a_{,}^2}b_{5,4}$$

$$=-\frac{75}{32}\frac{a}{a_{,}^2}b_{5,2}+\frac{3}{2}\frac{a^2}{a_{,}^3}b_{5,3}+\frac{9}{32}\frac{a}{a_{,}^2}b_{5,4}$$

Operating in the same way on all the terms in R multiplied by the squares of the eccentricities, we obtain finally the quantity

$$+\Sigma\left\{\frac{\{a^2e^2+a_{,}^2e_{,}^2\}}{32}b_{5,i}-\frac{3}{16}\frac{a}{a_{,}^2}\sin^2\frac{\bar{\imath}_{,}}{2}\left\{b_{5,i-1}+b_{5,i+1}\right\}\right.$$

$$\left.-\frac{3}{32}\frac{a}{a_{,}^2}(e^2+e_{,}^2)\left\{i\,b_{5,i-1}-i\,b_{5,i+1}\right\}\right\}\cos i\,t$$

$$+\Sigma\left\{\frac{\{2i+7\}}{64}\frac{a}{a_{,}^2}b_{5,i-1}+\frac{\{8i+13\}}{32}\frac{a^2}{a_{,}^3}b_{5,i}-\frac{\{18i+15\}}{64}\frac{a}{a_{,}^2}b_{5,i+1}\right\}e^2\cos(i\,t+2x)$$

$$+ \Sigma \left\{ - \frac{\{2i-3\}}{32} \frac{a}{a_{\prime}^2} b_{5,i-1} - \frac{i}{2} \frac{a^2}{a_{\prime}^3} b_{5,i} + \frac{\{18i-21\}}{32} \frac{a}{a_{\prime}^2} b_{5,i+1} \right\} e e_{\prime} \cos (it + x + z)$$

$$+ \Sigma \left\{ \frac{\{6i+7\}}{32} \frac{a}{a_{\prime}^2} b_{5,i-1} - \frac{\{6i+9\}}{32} \frac{a}{a_{\prime}^2} b_{5,i+1} \right\} e e_{\prime} \cos (it + x - z)$$

$$+ \Sigma \left\{ \frac{\{18i-15\}}{64} \frac{a}{a_{\prime}^2} b_{5,i-1} - \frac{\{8i-13\}}{32 a_{\prime}} b_{5,i} - \frac{\{2i-7\}}{64} \frac{a}{a_{\prime}^2} b_{5,i+1} \right\} e_{\prime}^2 \cos (it + 2z)$$

$$- \Sigma \frac{3}{8} \frac{a}{a_{\prime}^2} b_{5,i-1} \sin^2 \frac{\iota_{\prime}}{2} \cos (it + 2y)$$

The terms in R multiplied by the cubes of the eccentricities are equal to the preceding quantity multiplied by

$$- \frac{2a^2}{a_{\prime}^2} e \cos x + \frac{3a}{a_{\prime}} e \cos (t-x) + \frac{3a}{a_{\prime}} e_{\prime} \cos (t+z) - \frac{a}{a_{\prime}} e \cos (t+x)$$

$$- \frac{a}{a_{\prime}} e_{\prime} \cos (t-z) - 2 e_{\prime} \cos z ;$$

$$+ \left\{ - \frac{9}{8} \frac{\{a^2 e^2 + a_{\prime}^2 e_{\prime}^2\}}{a_{\prime}^2} + \frac{3}{8} \frac{a^2}{a_{\prime}^2} e^2 \cos 2x - \frac{3}{4} \frac{a}{a_{\prime}} \left\{ e^2 + e_{\prime}^2 + 2 \sin^2 \frac{\iota_{\prime}}{2} \right\} \cos t \right.$$

$$+ \frac{9}{16} \frac{a}{a_{\prime}} e^2 \cos (t + 2x) - \frac{9}{16} \frac{a}{a_{\prime}} e_{\prime}^2 \cos (t - 2z) + \frac{3}{16} \frac{a}{a_{\prime}} e^2 \cos (t - 2x)$$

$$+ \frac{3}{16} \frac{a}{a_{\prime}} e_{\prime}^2 \cos (t + 2z) + \frac{27}{8} \frac{a}{a_{\prime}} e e_{\prime} \cos (t - x + z) - \frac{9}{8} \frac{a}{a_{\prime}} e e_{\prime} \cos (t + x + z)$$

$$- \frac{9}{8} \frac{a}{a_{\prime}} e e_{\prime} \cos (t - x - z) + \frac{3}{8} \frac{a}{a_{\prime}} e e_{\prime} \cos (t + x - z) + \frac{3}{2} \frac{a}{a_{\prime}} \sin^2 \frac{\iota_{\prime}}{2} \cos (t + 2y)$$

$$\left. + \frac{3}{8} e_{\prime}^2 \cos 2z \right\}$$

$$\left\{ \Sigma \left\{ - \frac{a}{4a_{\prime}^2} b_{5,i-1} - \frac{a^2}{2a_{\prime}^3} b_{5,i} + \frac{3a}{4a_{\prime}^2} b_{5,i+1} \right\} e \cos (it + x) \right.$$

$$\left. + \Sigma \left\{ \frac{3}{4} \frac{a}{a_{\prime}^2} b_{5,i-1} - \frac{1}{2a_{\prime}} b_{5,i} - \frac{a}{4a_{\prime}^2} b_{5,i+1} \right\} e_{\prime} \cos (it + z) \right\}$$

$$+ \frac{1}{2a_{\prime}^3} \left\{ \frac{a^2 e^3}{4} \cos x - \frac{a^2 e^3}{4} \cos 3x + \frac{3}{2} a a_{\prime} e e_{\prime}^2 \cos (t - x) - a a_{\prime} \left\{ \frac{3}{4} e^3 + \frac{e e_{\prime}^2}{2} \right\} \cos (t + x) \right.$$

$$+ \frac{2 a a_{\prime}}{3} e^3 \cos (t + 3x) + \frac{a a_{\prime}}{12} e^3 \cos (t - 3x) + \frac{3 a a_{\prime}}{2} e_{\prime} e^2 \cos (t + z)$$

$$- \frac{9}{8} a a_{\prime} e^2 e_{\prime} \cos (t + 2x + z) - \frac{3}{8} a a_{\prime} e^2 e_{\prime} \cos (t - 2x + z)$$

$$- a\,a_{,}\left\{\frac{3}{4}\,e_{,}^{3} + \frac{e^{2}e_{,}}{2}\right\}\cos(t - z) + \frac{3}{8}\,a\,a_{,}e^{2}e_{,}\cos(t + 2x - z)$$

$$- \frac{9}{8}\,a\,a_{,}e\,e_{,}^{2}\cos(t - x - 2z) + \frac{3}{8}\,a\,a_{,}e\,e_{,}^{2}\cos(t + x - 2z) + \frac{2}{3}\,e_{,}^{3}\cos(t - 3z)$$

$$- \frac{3}{8}\,e\,e_{,}^{2}\cos(t - x + 2z) + \frac{e\,e_{,}^{2}}{8}\cos(t + x + 2z) + \frac{e_{,}^{3}}{12}\cos(t + 3z)$$

$$- 3\,a\,a_{,}e\sin^{2}\frac{\iota_{,}}{2}\cos(t + x - 2y) + a\,a_{,}e\sin^{2}\frac{\iota_{,}}{2}\cos(t - x - 2y)$$

$$- 3\,a\,a_{,}e_{,}\sin^{2}\frac{\iota_{,}}{2}\cos(t + z - 2y) + a\,a_{,}e_{,}\sin^{2}\frac{\iota_{,}}{2}\cos(t - z - 2y)$$

$$\left. + \frac{a_{,}^{2}e_{,}^{3}}{4}\cos z - \frac{a_{,}^{2}e_{,}^{3}}{4}\cos 3z\right\}.$$

$$\left\{\frac{b}{2}{}_{3,0} + b_{3,1}\cos t + b_{3,2}\cos 2t + \&c.\right.$$

+ terms independent of b.

Multiplying out, the coefficient of each term may be put in terms of $b_{5,i-2}$, $b_{5,i-1}$, $b_{5,i}$, $b_{5,i+1}$ and $b_{5,i+2}$.

The quantities $b_{1,0}$, $b_{1,1}$, from which all the other quantities b_3, b_5, &c. depend, may be obtained at once from Table IX. in the Exercices de Calc. Intégral, by M. LEGENDRE, vol. iii. See also vol. i. p. 171. of the same work.

$$\left(1 + \frac{a}{a_{,}}\right)b_{1,0} = \frac{4}{\pi}\int\frac{d\varphi}{\Delta} \qquad \left(1 + \frac{a}{a_{,}}\right)b_{1,1} = \frac{2}{\pi}\int - \frac{d\varphi\cos 2\varphi}{\Delta}$$

the integrals being taken from $\varphi = 0$ to $\varphi = \frac{1}{2}\pi$.

$$\Delta = \surd(1 - c^{2}\sin^{2}\varphi)$$

$c^{2} = \frac{4\,a\,a_{,}}{(a + a_{,})^{2}} = \frac{4\,\alpha}{(1 + \alpha)^{2}}$, α[*] being $= \frac{a}{a_{,}}$ as in the notation of the Méc. Cél.

$$b_{1,0} = \frac{4}{\pi(1 + \alpha)}\,F^{1} \qquad b_{1,1} = \frac{2}{\pi(1 + \alpha)}\left\{\frac{2}{c^{2}}\,(F^{1} - E^{1}) - F^{1}\right\}$$

In the theory of Jupiter disturbed by Saturn, $\alpha = {\cdot}54531725$; and hence in this instance if $c = \sin\theta$, $\theta = 72^{\circ}\ 53'\ 17''$.

By interpolation, I find from Table IX. p. 424,

$$F\,(72^{\circ}\ 53'\ 18'') = 2{\cdot}6460986$$

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[*] ρ in the notation of WOODHOUSE's Astronomy, vol. iii. p. 287.

and $b_{1,0} = 2{\cdot}180214$, which differs but slightly from the value of $b_{1,0}$ given by LAPLACE, viz. 2·1802348.

The equation

$$\lambda = \frac{h}{r^2}\left\{1 - \frac{1}{h}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t\right\}$$

or

$$\delta\lambda = \frac{2h}{r}\,\delta\,\frac{1}{r} - \frac{h}{r^2}\int\frac{\mathrm{d}R}{\mathrm{d}\lambda}\,\mathrm{d}t$$

appears to me to give numerical results more simply than that made use of by LAPLACE,

$$\delta\lambda = \frac{\dfrac{2r\delta r + \mathrm{d}r\,\delta r}{a^2 n\,\mathrm{d}t} + \dfrac{an}{\mu}\left\{3\int\!\!\int \mathrm{d}t\,\mathrm{d}'R + 2\int r\left(\dfrac{\mathrm{d}R}{\mathrm{d}r}\right)\mathrm{d}t\right\}}{\sqrt{1-e^2}}$$

See Théor. Anal. vol. i. p. 491.

When, however, that part of the inequality only is wanted which has a small coefficient in the denominator, as in the great inequality of Jupiter, the latter equation seems preferable, which thus reduces itself to

$$\delta\lambda = \frac{3an}{\mu\sqrt{1-e^2}}\int\!\!\int \mathrm{d}t\,\mathrm{d}'R$$

The apparent difference between the value of the coefficient given by this equation and the former, (see Phil. Trans. 1831, p. 290,) arises, no doubt, from part of the expression given by the former containing *implicitly* the same small quantity in the numerator.

It appears from the last Number of the Bulletin des Sciences Mathématiques, that M. CAUCHY, in a Memoir read before the Academy of Turin, has given "definite integrals which represent the coefficient of any given cosine in the development of *R*," by which means the calculation of any given inequality depending on a high power of the eccentricity is much facilitated. A similar method is alluded to by M. POISSON, Mémoires de l'Institut, vol. i. p. 50.

The reader is requested to make the following corrections.

Phil. Trans. 1830, p. 331, line 11, *for* $r^{\backprime}\left(\frac{\mathrm{d}R}{\mathrm{d}r^{\backprime}}\right)$; *read* $r^{\backprime}\,s\left(\frac{\mathrm{d}R}{\mathrm{d}r^{\backprime}}\right)$.

p. 334, line 14, *for* $s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda^{\backprime}}\right)\frac{\mathrm{d}r^{\backprime}}{\mathrm{d}\lambda^{\backprime}}$,

read $r^{\backprime}\,s\left(\frac{\mathrm{d}R}{\mathrm{d}r^{\backprime}}\right)-\left(\frac{\mathrm{d}R}{\mathrm{d}\lambda^{\backprime}}\right)\frac{\mathrm{d}s}{\mathrm{d}\lambda^{\backprime}}$.

p. 334, line 15, *for* $s\left(\frac{\mathrm{d}R}{\mathrm{d}s}\right)$, *read* $r^{\backprime}\,s\left(\frac{\mathrm{d}R}{\mathrm{d}r^{\backprime}}\right)$.

Phil. Trans. 1831, p. 234.

TABLE I.

Line.	Column.	*for*	*read*
8	7	27	− 27
8	10	......	40
17	4	34	− 34
18	62	92	98
35	4	58	55
65	3	77	− 80
146	9	163	−163

TABLE II.

Line.	Column.	*for*	*read*
27	8	7	− 7
34	4	17	− 17
40	10	......	8

Line.	Column.	*for*	*read*
55	4	......	35
58	4	35	......
77	3	65	......
80	3	......	− 65
139	3	......	25
141	3	......	28
143	3	......	31
144	1	......	63
145	1	......	64
163	9	146	−146
164	8	147	......
165	8	......	147

Addition to TABLE I. p. 277.

6	149	169	177

P. 271, *for* $s=\left\{z_{146}+\frac{e^2}{2}z_{150}+\frac{e^2}{2}z_{149}\right\}\gamma\sin y+\&c.$,

$$\textit{read}\ s=\left\{z_{146}+\frac{e^2}{2}z_{150}+\frac{e^2}{2}{}_{149}\right\}\gamma\sin y$$

$$+\left\{z_{147}+\frac{e^2}{2}z_{151}+\frac{e^2}{2}z_{153}-\frac{r_1}{2}\right\}\gamma\sin(2t-y)$$

$$+\left\{z_{148}+\frac{e^2}{2}z_{152}+\frac{e^2}{2}z_{154}+\frac{r_1}{2}\right\}\gamma\sin(2t+y)+\&c.$$

P. 273, line 7, *for* $-\frac{3r_1}{2}$, *read* $+\frac{3r_1}{2}$.

P. 8 (of the preceding paper), line 10, *for* $\frac{20}{27}ee_{\prime}\cos(x+z)$, *read* $\frac{20}{27}ee_{\prime}\cos(x-z)$.

P. 10, line 9, *for* $\frac{105}{16}\left\{r_3'+\lambda_3\right\}$, *read* $\frac{105}{16}e^2\left\{r_3'+\lambda_3\right\}$.

P. 12, line 2, *for* $\frac{9}{8}r_3'$, *read* $\frac{9}{8}e^2r_3'$.

——— 3, *for* $\frac{3}{8}r_3'$, *read* $\frac{3}{8}e^2r_3'$.

———13, *for* $\frac{26}{69}\left\{r_5'-\lambda_5\right\}$, *read* $\frac{26}{69}\left\{r_5'+\lambda_5\right\}$.

P. 13, at foot, *insert*

$$-\underset{[97]}{\frac{21}{16}\left\{r_5'+\lambda_5\right\}\gamma^2e_{\prime}^2\sin(2t-2z-2y)}-\underset{[99]}{\frac{3}{16}\left\{r_5'-\lambda_5\right\}\gamma^2e_{\prime}^2\sin(2t+2z-2y)}.$$

P. 14, line 13, *for* $\frac{3}{16}\left\{\frac{3}{2}r_3'-\frac{1}{2}\lambda_3\right\}+\frac{25.3}{8.2}\left\{r_5'-\lambda_5\right\}-\frac{5.3}{8.2}\left\{r_5'+\lambda_5\right\}$,

read $\frac{3}{16}e^2\left\{\frac{3}{2}r_3'-\frac{1}{2}\lambda_3\right\}+\frac{25.3}{8.2}e_{\prime}^2\left\{r_5'-\lambda_5\right\}-\frac{5.3}{8.2}e_{\prime}^2\left\{r_5'+\lambda_5\right\}$.

P. 15, line 3, *for* $\frac{5.3}{8.2}\left\{r_1'+\lambda_1\right\}$, *read* $\frac{5.3}{8.2}\left\{r_5'+\lambda_5\right\}$.

——— 4, *for* $\frac{5.3}{8.2}\left\{r_3'-\lambda_3\right\}$, *read* $\frac{5.3}{8.2}\left\{r_5'-\lambda_5\right\}$.

———13, *for* $\frac{20}{27}e^2\left\{r_3'-\lambda_3\right\}$, *read* $\frac{20}{27}\left\{r_3'-\lambda_3\right\}$.

P. 37, line 5, *for* $s_{147}=z_{147}+r_{\prime}=\cdot04617$, *read* $s_{147}=z_{147}-\frac{r_1}{2}=\cdot03536$.

P. 38, line 20, *for* retrograde, *read* direct.

II. *On the Tides.* *By* John William Lubbock, *Esq., V.P. and Treas. R.S.*

Read November 17, 1831.

WHEN I was lately at Paris, M. **Bouvard** kindly allowed me to copy some of the Observations made at Brest. Since my return to this country, the observations I obtained have been discussed by M. **Dessiou**, with regard to the principal inequality, or that which is independent of the parallaxes and declinations of the luminaries and depends solely on the moon's age, that is, on the time of her passage through the plane of the meridian.

The result is exhibited in the following Table.

Table showing the interval between the Moon's Transit and the time of High Water at Brest, from the Observations made there in the year 1816.

Time of Moon's Transit.	Interval Observed.	Moon's Transit.	Interval Observed.	Moon's Transit.	Interval Observed.	Moon's Transit.	Interval Observed.
h m	h m	h m	h m	h m	h m	h m	h m
0 0	3 47·8	3 0	3 7·8	6 0	2 52·8	9 0	4 9
0 30	3 43·5	3 30	3 4·9	6 30	3 2·2	9 30	4 9·9
1 0	3 36·6	4 0	2 58·4	7 0	3 18·2	10 0	4 9·5
1 30	3 23	4 30	2 50	7 30	3 33·5	10 30	4 6·9
2 0	3 15·5	5 0	2 48·5	8 0	3 46·4	11 0	4 2·5
2 30	3 10·9	5 30	2 49·5	8 30	4 0	11 30	3 54·6

It appears from this Table, that the establishment of that part of the port of Brest where these observations were made is $3^h\ 48^m$; the Annuaire for 1831 gives $3^h\ 33^m$ for that quantity. The constant $\lambda - \lambda_,$ may be taken about $1^h\ 20^m$, or 20° in space, being the value assigned to it by **Bernoulli**, but differing considerably from its value in the port of London, which is 2^h, or 30° in space. This result is important, as showing, unfortunately, that Tables of the Tides for London are not applicable to Brest by merely changing the establishment, that is, by adding a constant quantity, as has been supposed hitherto. The same remark applies of course generally to any distant ports.

The preceding Table gives also $\frac{m\,\Pi^3}{m_,\,\Pi_,^3} = \tan 18° 45'$ about, the logarithm of which quantity is 9·5307813. The determination of **Laplace**, by other means, of the same quantity is $9\cdot5385031 = \log \tan 19° 4'$. **Bernoulli** took this constant $= 9\cdot60286 = \log. \frac{2}{5}$, without explaining how he obtained it. **Newton** determined $\frac{m\,\Pi^3}{m_{,}\,\Pi_{,}^{3}} = \frac{1}{4\cdot4815}$, making the mass of the moon much too great.

$19° 4' - 18° 45' = 19'$ or $1^{m}\,16^{s}$ in time. A difference therefore of about one minute in the sum of the *intervals* when the moon passes the meridian at $4^{h}\,20^{m}$ and at $10^{h}\,20^{m}$ would remove altogether the discrepancy between my determination and that of **Laplace**.

The following **Table** shows the differences between the times of high water calculated by M. **Dessiou** with the constants $3^{h}\,47^{m}\cdot8$, $1^{h}\,20^{m}$ and 9·5307813, according to the formula Phil. Trans. 1831, p. 387, line 17, and the times given by observation, and also the differences between the times calculated (with the correct establishment) by means of the Table of **Bernoulli** given in the Annuaire for 1829, p. 40, and the same times given by observation.

Moon's Transit.	Observed.	Calculated with the constants above.	Error of Calculation.	Calculated from the Table in the Annuaire.	Error of Calculation.
h m	h m	h m	h	h m	m
12 0	3 47·8	3 47·8	0	3 47·8	0
0 30	3 43·5	3 40·7	− 2·8		
1 0	3 36·6	3 33·2	− 3·4		
1 30	3 23	3 25·6	+ 2·6		
2 0	3 15·5	3 18·1	+ 2·6	3 14·3	− 1·2
2 30	3 10·9	3 10·9	0		
3 0	3 7·8	3 4	− 3·8		
3 30	3 4·9	2 58	− 6·9		
4 0	2 58·4	2 53·1	− 5·3	2 45·8	−12·6
4 30	2 50	2 49·7	− 0·3		
5 0	2 48·5	2 48·5	0		
5 30	2 49·5	2 50·1	+ 0·6		
6 0	2 52·8	2 55·3	+ 2·5	2 45·3	− 7·3
6 30	3 2·2	3 4·7	+ 2·5		
7 0	3 18·2	3 18	− 0·2		
7 30	3 33·5	3 33·3	− 0·2		
8 0	3 46·4	3 47·1	+ 0·7	3 50·8	+ 4·4
8 30	4 0	3 58·3	− 1·7		
9 0	4 9	4 4·9	− 4·1		
9 30	4 9·9	4 7·7	− 2·2		
10 0	4 9·5	4 7·3	− 2·2	4 10·8	+ 0·7
10 30	4 6·9	4 4·5	− 2·4		
11 0	4 2·5	4 0·1	− 2·4		
11 30	3 54·6	3 54·4	− 0·2		

The agreement so far between theory and observation is not less remarkable than that at the London Docks which I have before noticed, (see Phil. Trans. 1831, p. 388). The irregularities in the errors given in the fourth column arise from the paucity of the observations employed.

As it would be of great importance to predict, if possible, any remarkably high tides which might take place, in order that precautions might be taken to avoid any disastrous consequences, I requested M. Dessiou to calculate, from the Tables in the Companion to the British Almanac* for 1831, the times and heights of high water at the London Docks corresponding to some remarkably high tides which have been observed, in order to see how nearly those Tables can be depended upon in extreme cases.

The following Table exhibits the results he obtained.

	Time of High Water.		Height of High Water.		Direction of the Wind.
	Observed.	Calculated.	Observed.	Calculated.	
	h m	h m	ft. in.	ft. in.	
1812. Oct. 21.	2 0 A.M.	2 10 A.M.	25 1	23 7½	NW
	2 10 P.M.	2 30 P.M.	25 8	23 10	NW
1821. Dec. 28.	3 45 A.M.	4 10 A.M.	23 10	22 9¾	SE
	4 15 P.M.	4 29 P.M.	25 10	22 8½	ESE
1824. Dec. 23.	3 10 A.M.	3 28 A.M.	25 11	22 5½	NW
	3 40 P.M.	3 46 P.M.	23 6	22 5¼	S
1827. Oct. 23.	11 45 P.M.	0 5 A.M.†	26 0	21 6½	NW
Nov. 1.	0 10 P.M.	0 25 P.M.	22 3	21 8½	NW

These results are extremely unsatisfactory; and I fear that it will happen sometimes, although but rarely, that a considerable error will occur in the calculated times and heights of high water, owing no doubt to gales of wind in the Channel or North Sea, or even perhaps in the Atlantic. The average error in using the Tables of the Companion, as M. Dessiou found by his calculations for the year 1826, (see Phil. Trans. 1831, p. 381,) in the time of high water is about 12^{m}, and in the height about 8 inches; this error however is more I believe to be attributed to the imperfection of the observations than to the inaccuracy of the Tables. The time is only recorded in the Dock books to the nearest five minutes.

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* In using these Tables the moon's transit should be equated or reduced to mean time before the corrections are applied. The example given in the Companion is therefore incorrect.

† November 1.

The Committee of the Astronomical Society, to whom the improvement of the Nautical Almanac was referred, having recommended the insertion in that work of a "Table of the mean time of high water at London Bridge for every day in the year, and also at the principal ports at the time of new and full moon," (see Report of the Committee of the Astronomical Society relative to the improvement of the Nautical Almanac, p. 14,)—without doubt that accuracy will be introduced into these calculations which has long been applied to all other astronomical phænomena.

In the open ocean the rise of the tide is so small that it is difficult to fix the time of high water, and the effect of the wind is so capricious, that it seems difficult to do more than to determine the establishment of the port; to which the mean of all the times of high water observed at any point of the lunation, will *in this case* afford a sufficient approximation. When this constant has been obtained at many places on the surface of the globe, the march of the great tide-wave will be ascertained, the numbers given on the map drawn by Mr. Walker, and which accompanies my former paper on this subject, may be rectified, and many anomalies which it now presents will no doubt disappear.

In narrow channels and archipelagoes the case is widely different: here the moon's age and even her parallax and declination have a perceptible influence; and if accuracy be required, all these circumstances, together with the period of the year, must be taken into account.

The observations which already exist would, if carefully discussed, furnish the means of determining the establishment of the port ($\lambda_{\prime}$), the fundamental hour of the port (λ), and the constant $\left(\frac{m\,\Pi^3}{m_{\prime}\,\Pi_{\prime}^3}\right)$, which contains implicitly the mass of the moon throughout the British Isles, and probably in many other places, as along the coast of France, at Madras, &c.* Having obtained these constants, Tables might be constructed, which by merely adding a given quantity would be sufficiently correct practically for a considerable extent of coast. These constants have been determined for the London Docks, and for the present time, with great precision, but the *establishment* is subject to change, and the determination of this quantity will probably require to be repeated after several years.

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* As is done in this paper for Brest.

With respect to the determination of the influence of the parallax and declination of the moon, it is desirable to employ more observations than I have done; I contented myself with about 5000, in order to spare M. Dessiou's time. A similar discussion of observations of the tides at Brest or some other favourable situation is greatly to be wished for, in order to ascertain how far these effects are the same in different ports.

The discussion of the observations of the times and heights of low water at the London Docks also remains, which I have been obliged to postpone.

The height of the water at any given time and place may be calculated when the requisite constants have been determined. At the London Docks the height of the water expressed in feet is

$$16·68 + 4·448 \left\{ \cos 2 (\theta_{,} - \lambda_{,}) + ·3788 \cos 2 (\theta - \lambda) \right\}$$

which formula affords results agreeing nearly with observation, and which may be compared with the curves given by Mr. Palmer, (see Phil. Trans. 1831, Part I.). According to this expression the mean rise of the tide is 12 ft. 3 in.

When $\lambda - \lambda_{,} = 0$, (that is, at the London Docks when the moon passes the meridian at 2 o'clock,) the curve in question is the *curve of sines*.

According to M. Daussy, (Mémoire sur les Marées des Côtes de France, Connaissance des Temps 1834,) the height of high water varies with the atmospheric pressure, being highest when the barometer is lowest. This paper did not come to my knowledge until after these pages were in the press; but the determination by M. Daussy of the establishment of the port of Brest coincides with that which I have given, namely $3^h\ 48^m$.

III. *On the Structure of the Human Placenta, and its Connexion with the Uterus.* *By* Robert Lee, *M.D. F.R.S. &c. Physician to the British Lying-in-Hospital.*

Read November 17, 1831.

In the year 1780 Mr. John Hunter presented a paper to the Royal Society, in which he laid claim to the discovery of the true structure of the placenta and its communication with the vessels of the uterus. The following is the history of the appearances which he observed in the dissection of a woman who had died undelivered near the full term of utero-gestation, and from which appearances his conclusions were drawn respecting the natural structure of these parts. The veins and arteries of the uterus having been injected, an incision was made through the parietes, at the anterior part where the placenta adhered to the internal surface. Between the uterus and placenta lay an irregular mass of injected matter, and from this mass regular pieces of the wax passed obliquely between it and the uterus, which broke off, leaving part attached to that mass; and on attentively examining the portions towards the uterus, they plainly appeared to be a continuation of the veins passing from it to this substance, which proved to be the placenta. Other vessels, about the size of a crow-quill, were seen passing in the same manner, although not so obliquely. These also broke on separating the placenta and uterus, leaving a small portion on the surface of the placenta; and on examination they were discovered to be continuations of the arteries of the uterus. The veins were next traced into the substance of what appeared placenta; but these soon lost the regularity of vessels, by terminating at once upon the surface of the placenta, in a very fine spongy substance, the interstices of which were filled with yellow injected matter. He then examined the arteries; and tracing them in the same manner towards the placenta, found that, having made a twisted or close spiral turn upon themselves, they were lost on its surface.

On cutting into the placenta, he discovered in many places of its substance yellow injection, and in others red, and in many others these two colours mixed. The substance of the placenta, now filled with injection, had nothing of the vascular appearance nor that of extravasation, but had a regularity in its form which showed it to be naturally of a cellular structure, fitted to be a reservoir for blood.

From these appearances Mr. Hunter infers, "that the arteries which are not immediately employed in conveying nourishment to the uterus go on towards the placenta, and proceeding obliquely between it and the uterus, pass through the decidua without ramifying. Just before entering the placenta, after making two or three close spiral turns upon themselves, they open at once into its spongy substance, without any diminution of size and without passing behind the surface as above described.

"The veins of the uterus appropriated to bring back the blood from the placenta, commence from this spongy substance by such wide beginnings, as are more than equal to the size of the veins themselves. These veins pass obliquely through the decidua to the uterus, enter its substance obliquely, and immediately communicate with the proper veins of the uterus. This structure of parts points at once to the nature of the blood's motion in the placenta. The blood detached from the common circulation of the mother moves through the placenta of the fœtus, and is then returned back into the course of the circulation of the mother to pass on to the heart*."

Dr. William Hunter's description of the vascular connexion between the uterus and placenta coincides with that of his brother: "for it seems incontestable (he observes) that the human placenta, like that of the quadruped, is composed of two distinct parts, though blended together; viz. an umbilical which may be considered as a part of the fœtus, and an uterine which belongs to the mother; that each of these parts has its peculiar system of arteries and veins, and its peculiar circulation, receiving blood by its arteries and returning it by its veins; that the circulation through these two parts of the placenta differs in the following manner:—in the umbilical portion the arteries terminate in the veins by a continuity of canal, whereas in the uterine portion there are intermediate cells, into which the arteries terminate, and from which the veins begin*."

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* Observations on certain Parts of the Animal Œconomy, by John Hunter, 1786: page 127.

It is a singular fact, that these celebrated anatomists should both have asserted their claims to the merit of what they supposed to be the discovery of the true structure of the human placenta, and its connexion with the uterus, and that their controversy on this subject should have loosened those bonds of affection which had united them together from their earlier years†.

NOORTWYCH, **RŒDERER**, and **HALLER**, had previously investigated this subject by injecting the blood-vessels of the gravid uterus: their researches however did not determine, in a satisfactory manner, that a vascular connexion exists between the uterus and cells in the placenta. The opinions of the **HUNTERS** were generally acquiesced in at the time they were promulgated, and their accuracy has not been called in question by any anatomist of reputation in this country for the last forty years.

In the communication which I have now the honour of presenting to the Royal Society, I propose to describe certain appearances which I have observed in the examination of six gravid uteri, and many placentæ expelled in natural labour, which seem to demonstrate that a cellular structure does not exist in the placenta, and that there is no connexion between this organ and the uterus by great arteries and veins.

If an incision be made through the parietes of the gravid uterus, where the placenta does not adhere, the membrana decidua will be observed lining the internal surface, and numerous minute blood-vessels and fibres passing from the inner membrane of the uterus to the decidua. At the circumference of the placenta, the membrana decidua separates from the chorion and amnion to pass between the uterus and placenta, and thus forms a complete membranous septum, which is interposed betwixt these organs. The chorion and amnion cover the fœtal surface of the placenta; and between these two membranes and the decidua lie the ramifications of the umbilical vein, and arteries subdivided to an almost indefinite extent, and connected together by white slen-

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[*] Anatomical Description of the Gravid Uterus and its Contents, by the late W. HUNTER, M.D. London, 1794: page 48.

[†] Their letters are preserved in the Archives of the Royal Society.

der filaments running in various directions. The placenta thus consists solely of a congeries of the umbilical vessels, covered on the fœtal surface by the chorion and amnion, and on the uterine surface by the deciduous membrane, and inclosed between these membranes; it adheres to the fundus, or some part of the uterus by innumerable flocculent fibres and vessels.

On detaching the placenta carefully from the uterus, the deciduous membrane is found to adhere so closely to the umbilical vessels which it covers, that it is impossible to remove it without tearing these vessels. With the fibres uniting the placental decidua to the uterus are mingled numerous small blood-vessels, proceeding from the inner membrane of the uterus to the decidua; and these vessels, though more numerous at the connexion of the placenta with the uterus, exist universally throughout the whole extent of the membrane. There is no vestige of the passage of any great blood-vessel, either artery or vein, through the intervening decidua, from the uterus to the placenta; nor has the appearance of the orifice of a vessel been discovered, even with the help of a magnifier, on the uterine surface of the placenta. This surface of the placenta deprived of the deciduous membrane presents a mass of floating vessels, its texture being extremely soft and easily torn; and no cells are discernible in its structure, by the minutest examination.

At that part of the surface of the uterus to which the placenta has been adherent, there are observable a great number of openings leading obliquely through the inner membrane of the uterus, and large enough to admit the point of the little finger: their edges are perfectly smooth, and present not the slightest appearance of having been lacerated by the removal of the placenta. In some places they have a semilunar or elliptical form, and in others they resemble a double valvular aperture. Over these openings in the inner membrane of the uterus, the placenta, covered by deciduous membrane, is directly applied, and closes them in such a manner that the maternal blood, as it flows in the uterine sinuses, cannot possibly escape either into the cavity of the uterus, or into the substance of the placenta. The above appearances on the inner surface of the uterus have been accurately represented by Rœderer; from whose work fig. 1. of Plate I. is taken.

When air is forcibly thrown either into the spermatic arteries or veins, the whole inner membrane of the uterus is raised by it; but none of the air passes

Fig. 2

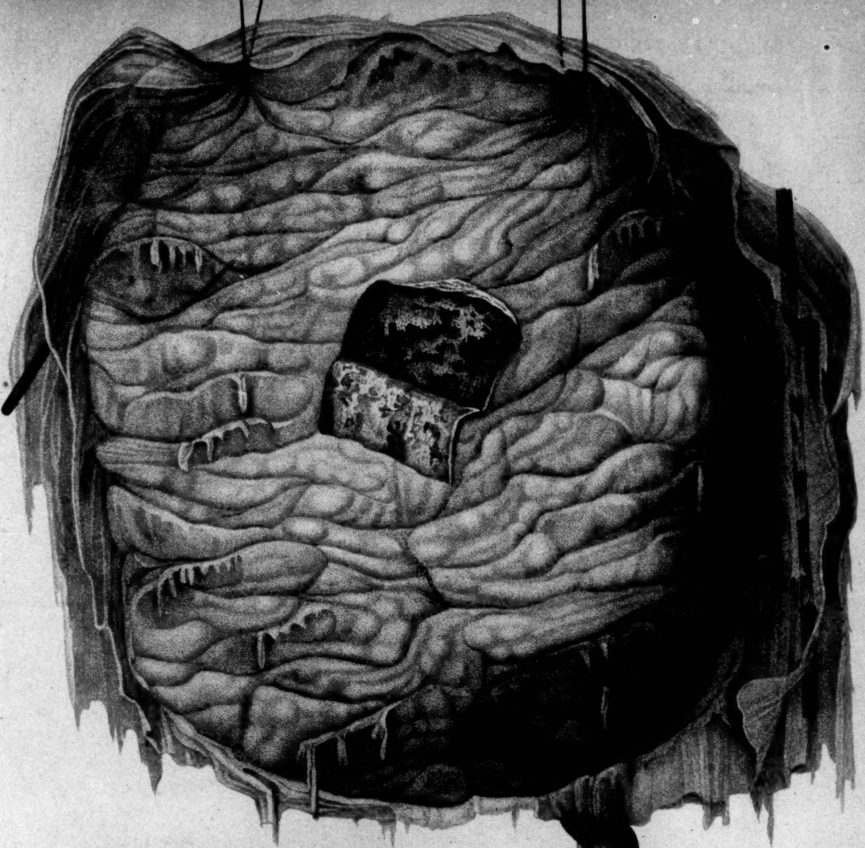
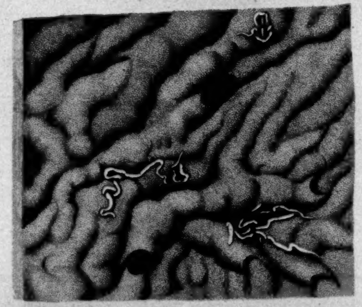


Fig. 1.



James Basire sculp.

across the deciduous membrane into the placenta, nor does it escape from the semilunar openings in the inner membrane of the uterus, until the attachment of the deciduous membrane to the uterus is destroyed. There are no openings in the deciduous membrane corresponding with these valvular apertures now described, in the internal membrane of the uterus. The uterine surface of the placenta is accurately represented in fig. 2. Plate I.

If a placenta be examined which has recently been separated from the uterus in natural labour, without any artificial force having been employed, its surface will be found uniformly smooth, and covered with the deciduous membrane; which could not be the case, did any large vessels connect it with the uterus. The placenta in a great majority of cases is also detached from the uterus after labour, with the least imaginable force; which would be impossible if a union by large blood-vessels, possessing the ordinary strength of arteries and veins, actually existed. Besides, a vascular connexion of such a kind would be likely to give rise, in every case, to dangerous hemorrhage subsequent to parturition, a circumstance not in accordance with daily experience.

NOORTWYCH, RŒDERER, HALLER, Dr. W. and Mr. J. HUNTER, and Dr. DONALD MONRO, do not appear to have examined the gravid uterus and its contents in the natural state of the parts, but after fluids had been forcibly injected into the hypogastric and spermatic arteries. The laceration of the deciduous membrane covering the orifices of the uterine sinuses followed this artificial process, as well as the formation of deposits of injection in the vascular structure of the placenta, giving rise to the deceptive appearance of cells. That this took place in the examinations made by RŒDERER* and MONRO†, does not admit of dispute; and the following facts render it more than probable that the HUNTERS were also misled, by the effects of artificial distention of the placenta, from the extravasation of the fluids forced into the uterine vessels.

In the course of last autumn, the preparations of the gravid uterus in the Hunterian Museum at Glasgow were examined at my request by Dr. NIMMO; and in none of them does it appear certain that any great blood-vessels pass from the uterus into cells in the placenta; but in many the deposits of injection, causing the appearance of cells, were observed evidently to be the result of extravasation. No preparation in the collection seems to have been expressly made for the purpose of proving or disproving the fact that the deciduous membrane passes over the uterine surface of the placenta; but in reference to preparation R. R. No. 139, it is observed by Dr. Nimmo that no vascular openings are visible in the membrane interposed between the uterus and placenta.

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* Icones Uteri humani, Observationibus illustratæ. J. G. RŒDERER, 1759.

† Essays and Observations, Physical and Literary, read before a Society in Edinburgh, 1754. vol. i.

No. 178. "is a small section of the uterus with the veins injected green, and broken off where they were entering the placenta." The surface of the injected matter is smooth; the edges of the openings defined and quite unlike ruptured vessels; their form in general elliptical, seeming as if they were holes cut in the side of a convolution.

No. 125. "A portion of uterus and placenta, the latter injected from uterine vessels." There is an opening which seems to be natural, corresponding to one of those in the uterus; but the majority of those whereby the injection has passed into the placenta seem to be mere lacerations.

No. 101. "A section of uterus with veins injected black, and the injected matter protruding by irregular plugs into the cavity of the uterus." The holes are semilunar and elliptical, with defined edges, and nothing resembling the continuation of vascular tubes to be seen.

R. R. 121. is described in the printed Catalogue as follows: "A small portion of placenta and uterus where the cells of the placenta have been injected from the veins of the uterus. The veins are seen very large, entering the substance of the placenta."

Dr. Nimmo makes the following observations on this specimen: "This preparation seems to be most in point. I would describe it differently. The cellular substance of the placenta has certainly been filled from the uterine vessels. These, however, instead of passing directly into the placenta, are distinctly seen applying their open mouths to the membrane of the placenta, where the injection in some instances stops. The membrane is thinner here than where no vessels are applied, consisting, so to describe it, of one layer, while a second layer covers all other parts. Where the injection has passed into the substance of the placenta, it has evidently been forced to the side between the layers, and found some weak point, whereby it has entered into and been diffused throughout the cellular texture of the placenta*."

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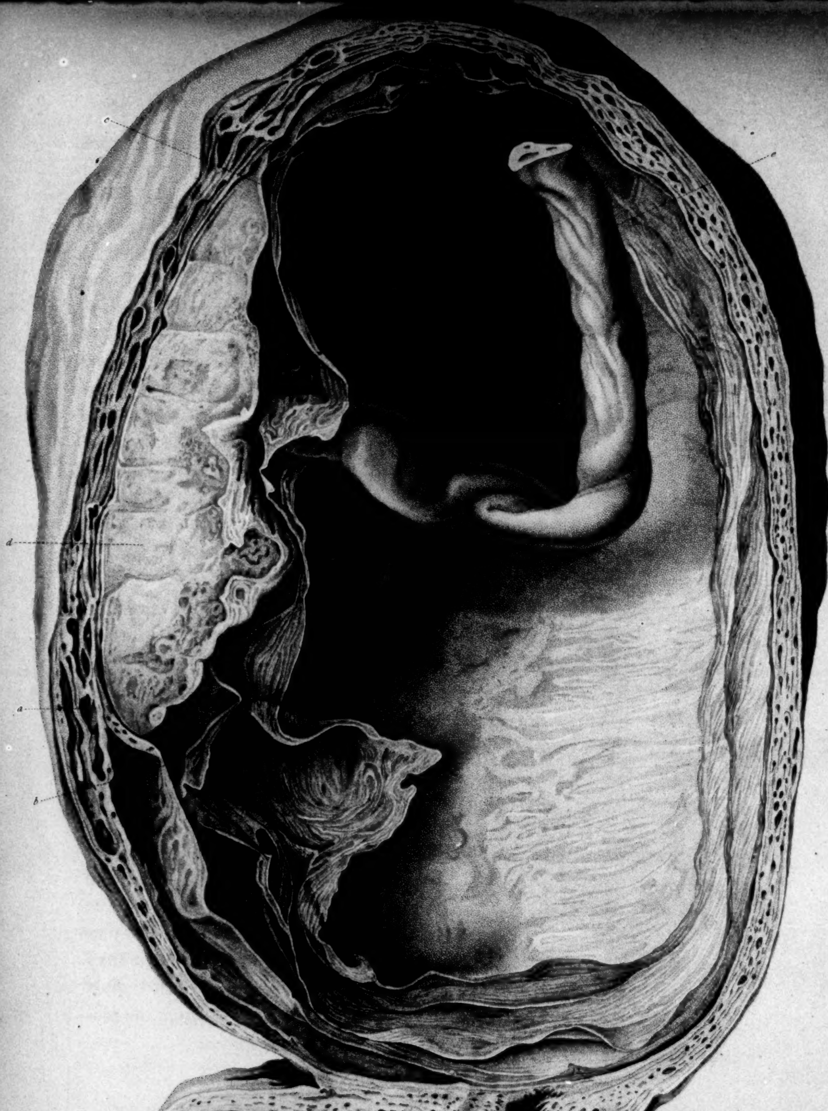
* My friend Samuel Broughton, Esq., F.R.S., during a recent visit to the Hunterian Museum at Glasgow, examined the preparations of the placenta and uterus at my request, and authorizes me to say that his observations fully confirm the accuracy of Dr. Nimmo's statements.


c
e
d
a
b

In the Museum of the Royal College of Surgeons of London, there is a preparation of the uterus with the placenta adhering to the inner surface, which is supposed to have been put up by Mr. HUNTER himself nearly fifty years ago. The vessels both of the uterus and placenta have been filled with injection, and the parietes of the uterus, placenta and membranes, have all been divided by a vertical section into two nearly equal portions. By permission of the Board of Curators, I have been enabled to examine one of these portions, and to have a drawing of it made. In the interstices of the muscular fibres I observed the veins of the uterus, which ran in great numbers towards the part where the placenta adhered. They were of an oval form, their long axes being in the long axis of the uterus. The muscular fibres ran longitudinally from the fundus to the os uteri. (Plate II.)

The deciduous membrane was everywhere covered with minute, tortuous blood-vessels proceeding from the inner surface of the uterus, and filled with injection. There was no appearance of vessels of any magnitude passing between the inner surface of the uterus and placenta; but flattened portions of injection were observed in this situation, having in many parts the form of thin layers, which had obviously escaped from the orifices of the uterine veins. Elsewhere the injection had lacerated the deciduous membrane, and formed deposits in the vascular part of the placenta.

The facts which have now been stated warrant, I think, the conclusion, that the human placenta does not consist of two parts, maternal and fœtal, that no cells exist in its substance, and that there is no communication between the uterus and placenta by large arteries and veins. The whole of the blood sent to the uterus by the spermatic and hypogastric arteries, except the small portion supplied to its parietes and to the membrana decidua by the inner membrane of the uterus, flows into the uterine veins or sinuses, and after circulating through them, is returned into the general circulation of the mother by the spermatic and hypogastric veins, without entering the substance of the placenta. The deciduous membrane being interposed between the umbilical vessels and the uterus, whatever changes take place in the fœtal blood, must result from the indirect exposure of this fluid, as it circulates through the placenta, to the maternal blood flowing in the great uterine sinuses.

Since the preceding paper was forwarded to the Secretary of the Royal Society, the following valuable communication has been received by the author from Mr. Owen, to whom portions of the gravid uterus and placenta were submitted for minute examination.

My dear Sir, Lincoln's Inn Fields, 17th November.

During the time you were examining the Hunterian preparation of the uterus and placenta in the Museum of the Royal College of Surgeons, your observations on the obscurity produced by the extravasated injection led me to think of some less objectionable mode of demonstrating the vascular communication between the uterus and placenta, if it existed; or of proving, more satisfactorily than the appearances you pointed out in that preparation seemed to do, that there was no such communication.

You have since afforded me the means, through the kindness of Mr. Alex. Shaw, of examining in the manner I wished, the anatomical relations between the placenta and uterus. This has been done by dissecting the parts under water before disturbing them, either by throwing forcibly foreign matter into the vessels, or by separating the placenta from the uterus to observe the appearances presented by the opposed surfaces,—a proceeding which if done in the air is liable to the objection of the possibility of having torn the vessels which were passing across, and the coats of which are acknowledged, by those who maintain the existence of such vessels, to be extremely delicate.

The mode, therefore, which was adopted to avoid these objections, was to fix under water in an apparatus used for dissecting mollusca, &c., a section of the uterus and placenta, and, commencing the dissection from the outside, to remove successively and with care, the layers of fibres, and trace the veins as they pass deeper and deeper in the substance of the uterus in their course to the deciduous membrane; in which situation as the thinnest pellicle of membrane is rendered distinct by being supported in the ambient fluid, I naturally hoped in this way to see the coats of the veins continued into the deciduous membrane and placenta, and to be able to preserve the appearance in a preparation, if it actually existed in nature. But in every instance the vein, having reached the inner surface of the uterus, terminated in an open mouth on that aspect; the peripheral portion of the coat of the vein, or that next the uterus, ending in a well-defined and smooth semicircular margin, the central part adhering to, and being apparently continuous with, the decidua.

In the course of this dissection I observed that where the veins of different planes communicated with each other, the central portion of the parietes of the superficial vein invariably projected in a semilunar form into the deeper-seated one; and where (as was frequently the case, and especially at the point of termination on the inner surface) two or even three veins communicated with a deeper-seated one at the same point, these semilunar edges decussated each other so as to allow only a very small part of the deep-seated vein to be seen. I need not observe to you how admirably this structure is adapted to ensure the effect of arresting the current of blood through these passages, upon the contraction of the fibres with which they are everywhere surrounded.

On another portion of the same uterus and placenta, (which were removed from a woman who died at about the fifth month of utero-gestation,) I commenced the examination under water by turning the placenta and deciduous membrane from the inner surface of the uterus. In this way the small tortuous arteries that enter the deciduous membrane were readily distinguishable, though not filled with injected matter; and as it was an object to avoid unnecessary force in the process of separation, they were cut through, though they are easily torn from the decidua. But with respect to the veins, they invariably presented the same appearances as were noticed in the first dissection, terminating in open semicircular orifices, which are closed by the apposition of the deciduous membrane and placenta. This membrane is, however, certainly thinner opposite these orifices than elsewhere; and in some places appeared to be wanting, or adhering to the vein was torn up with it; but in these cases the minute vessels of the placenta only appeared, and never any indication of a vascular trunk or cell commensurate with the size of the vein whose terminal aperture had been lifted up from the part.

The preparation which accompanies this letter shows the termination of a vein on the inner surface of the uterus, and an artery of the decidua cut through, with the corresponding appearances on the surface of the placenta,—also the valvular mode in which the veins communicate together in the substance of the uterus.

I remain yours very truly,

Richard Owen.

Explanation of the PLATES.

PLATE I.

Fig. 1.—**Represents the openings in the inner membrane of the uterus, where the placenta had adhered.**

Fig. 2.—**A view of the uterine surface of the placenta, covered by the membrana decidua.**

PLATE II.

A section of the gravid uterus, placenta, and membranes.

a. **Uterine sinuses injected.**
b. **The membrana decidua passing between the uterus and placenta.**
c. **The chorion and amnion passing over the fœtal surface of the placenta.**
d. **The vessels which compose the placenta.**
e. **The umbilical chord.**

IV. *On an inequality of long period in the motions of the Earth and Venus.*

By **George Biddell Airy**, *A.M., F.R. Ast. Soc., F.G.S., late Fellow of Trinity College, Cambridge, and Plumian Professor of Astronomy and Experimental Philosophy in the University of Cambridge. Communicated by Sir* J. F. W. **Herschel**, *F.R.S. &c. &c. &c.*

Read November 24, 1831.

In a paper "On the corrections of the elements of **Delambre**'s Solar Tables," published in the Philosophical Transactions for 1828, I stated that the comparison of the corrections in the epochs of the sun and the sun's perigee given by late observations, with the corrections given by the observations of the last century, appeared to indicate the existence of some inequality not included in the arguments of those Tables. As soon as I had convinced myself of the necessity of seeking for some inequality of long period, I commenced an examination of the mean motions of the planets, with the view of finding one whose ratio to the mean motion of the earth could be expressed very nearly by a proportion whose terms were small: and I did not long seek in vain.

It is well known that the appearances of Venus recur in very nearly the same order every eight years: and therefore some multiple of the periodic time of Venus is nearly equal to eight years. It is easily seen that this multiple is thirteen: and consequently eight times the mean motion of Venus is nearly equal to thirteen times the mean motion of the Earth. According to **Laplace**, (Méc. Cél. liv. vi. chap. 6.) the mean annual motion of Venus is $650^{g}\cdot198$; that of the Earth $399^{g}\cdot993$. Hence

8 × mean annual motion of Venus . . . . = $5201^{g}\cdot584$

13 × mean annual motion of the Earth . . = $5199\cdot909$

Difference . . . . = $1\cdot675$

The difference is about $\frac{1}{240}$ of the mean annual motion of the Earth; and it implies the existence of an inequality whose period is about 240 years. No term has yet been calculated whose period is so long with respect to the periodic time of the planets disturbed [*]. The probability that there would be found some sensible irregularity depending on this term, may be estimated from this consideration; that in integrating the differential equations, this term receives a multiplier of $3 \times 13 \times (240)^2$, or about 2,200,000.

On the other hand, the coefficient of this term is of the fifth order (with regard to the excentricities and inclinations of the orbits). The excentricities of both orbits are small. And it is remarkable that in the present position of the perihelia, the terms which would otherwise produce a large inequality destroy each other almost exactly. The inclination however is not so small; and upon this the existing inequality depends principally for its magnitude.

The value of the principal term, calculated from the theory, I gave in a postscript to the paper above cited. I propose in the present memoir to give an account of the method of calculation, and to include other terms which are necessarily connected with the principal inequality.

PART I.

PERTURBATION OF THE EARTH'S LONGITUDE AND RADIUS VECTOR.

SECTION 1.

Method adopted for this investigation.

1. The motion of a disturbed planet may be represented by supposing it to move, according to the laws of undisturbed motion, in an ellipse whose dimensions and position are continually changing: the epoch of the planet's mean longitude at the origin of the time being also supposed to change. Putting a for the semi-axis major; e for the excentricity; ϖ for the longitude of perihelion; n for the mean motion in longitude in a unit of time; ϵ for the epoch, or the mean longitude when $t = 0$; (all which are variable): m for the mass of the planet (Venus); μ for the sum of the masses of the sun and planet; and the same letters with accents for the same quantities relative to another planet (the

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[*] The period of the long inequality of Saturn is only about thirty times as great as the periodic time of Saturn.

Earth); the variation of the elements of the second planet's orbit will be given by the following equations:

$$\frac{da'}{dt} = -\frac{2n'a'^2}{\mu'} \cdot \frac{dR}{d\epsilon'}$$

$$\frac{dn'}{dt} = +\frac{3n'^2a'}{\mu'} \cdot \frac{dR}{d\epsilon'}$$

$$\frac{de'}{dt} = -\frac{n'a'}{\mu'e'}(1-e'^2)\frac{dR}{d\epsilon'} + \frac{n'a'(1-e'^2)^{\frac{1}{2}}}{\mu'e'}\left(\frac{dR}{d\epsilon'} + \frac{dR}{d\varpi'}\right)$$

$$\frac{d\varpi'}{dt} = -\frac{n'a'}{\mu'e'}(1-e'^2)^{\frac{1}{2}} \cdot \frac{dR}{de'}$$

$$\frac{d\epsilon'}{dt} = -\frac{3n'^2a'}{\mu'} \cdot \frac{dR}{d\epsilon'} \cdot t + \frac{2n'a'^2}{\mu'} \cdot \frac{dR}{da'} - \frac{n'a'}{\mu'e'}\left((1-e'^2)^{\frac{1}{2}} - (1-e'^2)\right)\frac{dR}{de'}$$

where R or $\frac{m(x'x+y'y)}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{m}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}$ is expanded in terms depending on the mean motions of the two planets. These expressions are true only on the supposition that the *actual* orbit of m' is in the plane of xy, or is so little inclined that the square of the inclination may be neglected. The values of a', e', &c. on the right-hand side of the equations ought in strictness to be the true variable values. But it will in general be sufficiently accurate to put for e' the value E which it had near the time for which the investigation is made, and to consider it as constant: or at any rate the expression $E + Ft$, where F is the mean value of its increase when $t = 0$: and similarly for the others. Determining thus the values of $\frac{da'}{dt}$, $\frac{de'}{dt}$, &c. and from them those of a', e', &c., they are to be substituted in the expressions

$$r' = a'\left\{1 + \frac{1}{2}e'^2 + \left(-e' + \frac{3}{8}e'^3 - \&c.\right)\cos(n't + \epsilon' - \varpi') \right.$$
$$\left. + \left(-\frac{1}{2}e'^2 + \&c.\right)\cos(2n't + 2\epsilon' - 2\varpi') + \&c.\right\}$$

$$v' = n't + \epsilon' + \left(2e' - \frac{1}{4}e'^3 + \&c.\right)\sin(n't + \epsilon' - \varpi')$$
$$+ \left(\frac{5}{4}e'^2 - \frac{11}{24}e'^4 + \&c.\right)\sin(2n't + 2\epsilon' - 2\varpi') + \&c.$$

and the true values of the radius vector and longitude are obtained.

2. When (as in the present instance) the inequality is so small that we may be satisfied with the principal part of it, we may in the expressions omit the powers of e'. Thus we have

$$\frac{d\,a'}{dt} = -\frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\epsilon'}$$

$$\frac{d\,n'}{dt} = +\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\epsilon'}$$

$$\frac{d\,e'}{dt} = +\frac{n'\,a'}{\mu'\,e'} \cdot \frac{d\,\mathrm{R}}{d\,\varpi'}$$

$$\frac{d\,\varpi'}{dt} = -\frac{n'\,a'}{\mu'\,e'} \cdot \frac{d\,\mathrm{R}}{d\,e'}$$

$$\frac{d\,\epsilon'}{dt} = -\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\epsilon'} \cdot t + \frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,a'} - \frac{1}{2} \cdot \frac{n'\,a'\,e'}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,e'}$$

3. Hitherto this method has been actually used (I believe) only for the calculation of secular variations. But it can be applied with great advantage in almost every case: and in the instance before us it is particularly convenient, as it requires only the development of a single term. For if in the development of R we take the terms depending on cos $\{13\,n't - 8\,n\,t + \mathrm{A}\}$, whose coefficient is of the 5th order, it will be found that $\frac{d\,a'}{dt}$, $\frac{d\,n'}{dt}$, and $\frac{d\,\epsilon'}{dt}$, are of the 5th order, $\frac{d\,e'}{d\,t}$ of the 4th order, and $\frac{d\,\varpi'}{d\,t}$ of the 3rd order. Integrating these expressions, and substituting them in the formula for v', there will be produced terms of the forms $\frac{p}{(13\,n' - 8\,n)^2}$ sin $\{13\,n't - 8\,n\,t + \mathrm{B}\}$ and $\frac{q}{13\,n' - 8\,n}$ sin $\{12\,n't - 8\,n\,t + \mathrm{C}\}$, where p is of the 5th, and q of the 4th order. And a little examination will show that no other argument will produce terms of the same or of a lower order, which are divided by the small quantity $13\,n' - 8\,n$: inasmuch as this divisor is introduced only by integration of the expressions for $\frac{d\,a'}{d\,t}$, &c. Our object then at present is to select in the development of R all the terms of the form A cos $\{13\,n'\,t - 8\,n\,t + \mathrm{B}\}$. And as the inequality which we are seeking will probably be small, we may confine ourselves to those terms in which the order of the coefficient is the lowest possible: that is, to terms of the 5th order.

SECTION 2.

On the abridgement which the development admits of, and the notation which it permits us to use.

4. Let θ be the longitude of the node of the orbit of m (Venus), and φ its inclination: the orbit of m' (the Earth) being supposed to coincide with the plane of xy. Let v, the longitude of m, be measured* by adding the angular distance of m from its node to the longitude of the node. Then $v-\theta$ is the distance of m from the node. Let r be the true radius vector of m: then

$$x' = r' . \cos v'$$

$$y' = r' . \sin v'$$

$$x = r\,\{\cos(v-\theta) . \cos\theta - \sin(v-\theta) . \sin\theta . \cos\varphi\}$$

$$y = r\,\{\cos(v-\theta) . \sin\theta + \sin(v-\theta) . \cos\theta . \cos\varphi\}$$

$$z = r . \sin(v-\theta) . \sin\varphi$$

Substituting these, the expression for R becomes

$$\frac{m\,r'}{r'^3}\left\{\cos(v'-\theta) . \cos(v-\theta) + \cos\varphi . \sin(v'-\theta) . \sin(v-\theta)\right\}$$

$$-\frac{m}{\sqrt{\left\{r'^2 - 2\,r'\,r\left(\cos(v'-\theta) . \cos(v-\theta) + \cos\varphi . \sin(v'-\theta) . \sin(v-\theta)\right) + r^2\right\}}}$$

in which it must be remarked that r and v, when expressed in terms of t, will not involve the constants θ and φ. This may be changed into

$$\frac{m\,r'}{r'^3}\left\{\cos(v'-v) - \sin^2\frac{\varphi}{2} . \cos(v'-v) + \sin^2\frac{\varphi}{2}\cos(v'+v-2\,\theta)\right\}$$

$$-\frac{m}{\sqrt{\left\{r'^2 - 2\,r'\,r . \cos(v'-v) + r^2 + 2\,r'\,r . \sin^2\frac{\varphi}{2}\cos(v'-v) - 2\,r'\,r . \sin^2\frac{\varphi}{2}\cos(v'+v-2\,\theta)\right\}}}$$

or,

$$\frac{m\,r'}{r'^3}\cos(v'-v) - \frac{m}{\sqrt{\{r'^2 - 2\,r'\,r . \cos(v'-v) + r^2\}}}$$

$$+ \sin^2\frac{\phi}{2}\,\{\cos(v'-v) - \cos(v'+v-2\,\theta\} . \left\{\frac{m\,r'\,r}{\{r'^2 - 2\,r'\,r . \cos(v'-v) + r^2\}^{\frac{3}{2}}} - \frac{m\,r'}{r'^3}\right\}$$

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* ϖ, the longitude of the perihelion of m, must be measured in the same manner.

$$-\frac{3}{2}\sin^4\frac{\phi}{2}\left\{\cos(v'-v)-\cos(v'+v-2\theta)\right\}^2 . \frac{m r' r}{\{r'^2-2r'r.\cos(v'-v)+r^2\}^{\frac{5}{2}}}$$

$+$ &c.

5. The first line of this may be expanded in the form

$$-m\left\{\frac{1}{2}\Gamma_{\frac{1}{2}}^{(0)}+\Gamma_{\frac{1}{2}}^{(1)}\cos(v'-v)+\Gamma_{\frac{1}{2}}^{(2)}\cos(2v'-2v)+\&c.\right\}$$

where $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, &c., are functions of r' and r. We must then express r' and r in terms of $n't$ and nt, and must substitute these values in $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, $\Gamma_{\frac{1}{2}}^{(2)}$, &c. and must express v' and v in terms of $n't$ and nt; and on multiplying the respective expressions we shall have the development necessary for our method.

6. Now upon expressing r' in terms of $n't$, the following remarkable law always holds: The index of the term of lowest order in the coefficient of such an argument as $\cos(pn't+\text{A})$, is p. The same is true with regard to the development of r, v', and v.

7. Now such a term as $\text{A}\cos\{13n't-8nt+\text{B}\}$ can be produced only by the multiplication of $\frac{\cos}{\sin}\left(kn't-knt+k\epsilon'-k\epsilon\right)$, (from the first term in the development of $\cos(kv'-kv)\big)$, with $\frac{\cos}{\sin}\overline{(13\backsim k)(n't+\epsilon'-\varpi')}$ and $\frac{\cos}{\sin}\overline{(8\backsim k)(nt+\epsilon-\varpi)}$ $\Big($occurring in the development of $kv'-kv$, or of $\Gamma_{\frac{1}{2}}^{(k)}\Big)$. The largest term in the coefficient, according to the rule just explained, will be of the order whose index is the sum of $13\backsim k$ and $8\backsim k$. Now if k be < 8, as for instance if k be 7, the index of the order is $6+1=7$, or the term is of the 7th order, and therefore is to be rejected. And if k be > 13, as for instance if $k=14$, the index of the order is $1+6=7$, and the term is to be rejected. But if k be 8, or 13, or any number between them, as for instance 10, then the order of the term is $3+2=5$, and the term is to be kept. It appears therefore that the only terms which we shall have occasion to develope, are $\Gamma_{\frac{1}{2}}^{(8)}.\cos(8v'-8v)$, $\Gamma_{\frac{1}{2}}^{(9)}.\cos(9v'-9v)$, &c. as far as $\Gamma_{\frac{1}{2}}^{(13)}.\cos(13v'-13v)$ inclusively.

8. Supposing then k to be not less than 8 nor greater than 13, the term ${}^{\cos}_{\sin}\left(kn't - knt + k\varepsilon' - k\varepsilon\right)$ must be multiplied by ${}^{\cos}_{\sin}\left(\overline{(13-k)\,(n't+\varepsilon'-\varpi')} + \overline{(k-8)\,(nt+\varepsilon-\varpi)}\right)$ in order to produce a term of the form $A\cos(13\,n't - 8\,nt + B)$ whose coefficient is of the 5th order. The latter factor must have arisen from the product of two such terms as $e'^{\,13-k}\,.\,{}^{\cos}_{\sin}\,\overline{(13-k)\,(n't+\varepsilon'-\varpi')}$ and $e^{k-8}\,.\,{}^{\cos}_{\sin}\,\overline{(k-8)\,(nt+\varepsilon-\varpi)}$. The expansion of such a product will always produce two terms, one of which has for argument the sum of the arguments of the factors, and the other has the difference of the same arguments. The point to which I wish particularly to call the attention of the reader is this: The term of the product depending on the *sum* of the arguments is the only one which is useful to us. For instance; the product of $e'^2\,.\sin 2\,(n't+\varepsilon'-\varpi')$ and $e^3\,.\sin 3\,(nt+\varepsilon-\varpi)$ will be $-\frac{1}{2}e'^2e^3\,.\cos(2n't+3\,nt+2\varepsilon'+3\varepsilon-2\varpi'-3\varpi)$ $+\frac{1}{2}e'^2e^3\,.\cos(2\,n't-3\,nt+2\,\varepsilon'-3\varepsilon-2\,\varpi'+3\,\varpi)$; the combination of the first term with $\cos(11\,n't-11\,nt+11\,\varepsilon'-11\,\varepsilon)$ will produce a term of the form $A\cos(13\,n't-8\,nt+B)$ whose coefficient is of the 5th order: the second term will not produce a term of that form. We might choose terms, as $e'\,.\sin(n't+\varepsilon'-\varpi')$ and $e^6\,.\sin 6\,(nt+\varepsilon-\varpi)$ such that the part of the product depending on the difference of the arguments, or $\frac{1}{2}e'e^6\,.\cos(n't-6\,nt+\varepsilon'-6\varepsilon-\varpi'+6\,\varpi)$ combining with such a term as $\cos(14\,n't-14\,nt+14\,\varepsilon'-14\,\varepsilon)$, would produce a term of the form required: but its coefficient would not be of the 5th order. It is equally necessary to remark that, in multiplying the term thus selected by ${}^{\cos}_{\sin}\left(kn't-knt+k\varepsilon'-k\varepsilon\right)$, we again preserve only that part of the product depending on the *sum* of the arguments.

9. On the circumstance that, in taking the product of two circular functions, we have to retain only the term whose argument is the sum of the arguments, depends the principle of our notation. For whenever (in an advanced stage of the operations) such a term as ${}^{\cos}_{\sin}\left(2\,n't+3\,nt+2\,\varepsilon'+3\,\varepsilon-2\,\varpi'-3\,\varpi\right)$ occurs, we shall know that, being formed in accordance with this rule, it must

have arisen from the product of $e'^2 \frac{\cos}{\sin}\left(2n't+2\epsilon'-2\varpi'\right)$ and $e^3 \frac{\cos}{\sin}\left(3nt+3\epsilon-3\varpi\right)$; its coefficient therefore can only be $e'^2 e^3$. And conversely, from seeing this coefficient, we should be certain that the argument would be $2\,(n't+\epsilon'-\varpi') + 3\,(nt+\epsilon-\varpi)$. Instead therefore of writing

$$e'^2 e^3 . \cos\,(2\,n't + 3\,nt + 2\,\epsilon' + 3\,\epsilon - 2\,\varpi' - 3\,\varpi)$$

we might simply write

$$e'^2 e^3 . \cos$$

omitting the argument entirely. But it will be found more convenient to retain the figures in the argument, writing it thus,

$$e'^2 e^3 . \cos\,(2+3)$$

the first figure being always appropriated to the accented argument. And when this term is multiplied by $\cos\,(11\,n't - 11\,nt + 11\,\epsilon' - 11\,\epsilon)$ or $\cos\,(11-11)$, we may write down the result

$$\tfrac{1}{2}\,e'^2 e^3 . \cos\,(13-8)$$

without any fear of mistake. For we know that the argument must have been produced by adding $2\,(n't+\epsilon'-\varpi')$, $3\,(nt+\epsilon-\varpi)$, and $11\,(n't-nt+\epsilon'-\epsilon)$, and thus when a result is obtained the term can be filled up.

10. If we examine the second line in the last expression of (4), it is easily seen that $\sin^2 \frac{\varphi}{2}$, a quantity of the second order (considering $\sin \frac{\varphi}{2}$ as of the same order with e' and e) enters as multiplier into two terms: of which the first, or $\sin^2 \frac{\varphi}{2} . \cos\,(v'-v)$, when developed will have in every term one part of the argument produced by a subtraction; and therefore, when combined with the expansion of the term multiplying it, will produce terms $\cos\,(13-8)$ of the 7th order at lowest; the first term therefore is useless. But the second, or $-\sin^2 \frac{\varphi}{2} . \cos\,(v'+v-2\,\theta)$, is exactly analogous to $e^2 \cos\,(v'+v-2\,\varpi)$, which would arise from the product of $e^2 \cos\,(2v-2\,\varpi)$ and $\cos\,(v'-v)$, and to which all the preceding remarks would apply; and examination would show that in the development of this term, in which products of $\sin^2 \frac{\varphi}{2}$ with powers of e' and e

will occur, the same rule must be followed, namely, that the only useful terms in the products are those in which the arguments are added. And whenever $\sin^2\frac{\phi}{2}$ occurs in the coefficient, -2θ occurs in the argument; so that there will be no possibility of mistake in using the notation described in (9).

11. On examining the third line in the last expression of (4), it will be seen in the same manner that the only part of $-\frac{3}{2}\sin^4\frac{\phi}{2}\left\{\cos(v'-v)-\cos(v'+v-2\theta)\right\}^2$ to be preserved is $-\frac{3}{4}\sin^4\frac{\phi}{2}\,.\cos(2v'+2v-4\theta)$. The same remarks apply to this term as to the last; and for a similar reason the notation of (9) may be used without fear of mistake.

12. By the use of this notation we may in some instances materially shorten our expressions. For instance, we might have the terms

$$\begin{aligned}&F\,e'e^4\,.\cos(n't+4nt+\epsilon'+4\epsilon-\varpi'-4\varpi)\\&+G\,e'e^2\sin^2\frac{\phi}{2}\,.\cos(n't+4nt+\epsilon'+4\epsilon-\varpi'-2\varpi-2\theta)\\&+H\,e'\sin^4\frac{\phi}{2}\,.\cos(n't+4nt+\epsilon'+4\epsilon-\varpi'-4\theta)\end{aligned}$$

All this would be expressed without the possibility of mistake by the following term,

$$\left(F\,e'e^4+G\,e'e^2\sin^2\frac{\phi}{2}+H\,e'\sin^4\frac{\phi}{2}\right).\cos(1+4).$$

The utility of such abridgments, and the quantity of disgusting labour which they spare, can be conceived only by those who have gone through the drudgery of performing the actual operation.

13. It is only necessary to add that when we have, for the coefficient of a cosine or sine, a series proceeding by powers of e', e, $\sin^2\frac{\phi}{2}$, &c. we may always neglect all after the lowest power. For instance, the correct expression for v is

$$\begin{aligned}&nt+\epsilon\\&+\left(2e-\frac{1}{4}e^3+\frac{5}{96}e^5-\&c.\right)\sin(nt+\epsilon-\varpi)\end{aligned}$$

$$+\left(\frac{5}{4}e^2-\frac{11}{24}e^4+\&c.\right)\sin(2nt+2\epsilon-2\varpi)$$

$$+\left(\frac{13}{12}e^3-\frac{43}{64}e^5+\&c.\right)\sin(3nt+3\epsilon-3\varpi)$$

$$+\left(\frac{103}{96}e^4-\&c.\right)\sin(4nt+4\epsilon-4\varpi)$$

$$+\left(\frac{1097}{960}e^5-\&c.\right)\sin(5nt+5\epsilon-5\varpi)$$

$$+\&c.$$

but for our purposes it will be sufficient to take $v=(0+1)+2e\,.\sin(0+1)$ $+\frac{5}{4}e^2\,.\sin(0+2)+\frac{13}{12}e^3\,.\sin(0+3)+\frac{103}{96}e^4\,.\sin(0+4)+\frac{1097}{960}e^5\,.\sin(0+5)$. For none of the terms can be of any use to us till they are multiplied, so that the largest term of the coefficient is of the 5th order; and then all the other parts will be of a higher order.

14. Putting f for $\sin\frac{\phi}{2}$, it will be seen that (in conformity with the remarks in this section), the terms of R to be developed are

$$-\frac{m}{\sqrt{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}}}$$

$$-f^2\,\frac{mr'r\,.\cos(v'+v-2\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{3}{2}}}$$

$$-\tfrac{3}{4}f^4\,.\,\frac{mr'^2r^2\,.\cos(2v'+2v-4\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}$$

SECTION 3.

Expansion of cos $(kv'-kv)$, *to the fifth order.*

15. By (13) the value of $kv'-kv$ is

$$(k-k)$$

$$+2ke'\,.\sin(1+0)-2ke\,.\sin(0+1)\;.\;.\;.\;.\;.\;.\;.\;.\;.\qquad(\text{A})$$

$$+\frac{5}{4}ke'^2\,.\sin(2+0)-\frac{5}{4}ke^2\,.\sin(0+2)\;.\;.\;.\;.\;.\;.\;.\qquad(\text{B})$$

$$+\frac{13}{12}k\,e'^{3}\,.\sin(3+0)-\frac{13}{12}k\,e^{3}\,.\sin(0+3)\quad.\quad.\quad.\quad.\quad.\quad.\quad(\mathrm{C})$$

$$+\frac{103}{96}k\,e'^{4}\,.\sin(4+0)-\frac{103}{96}k\,e^{4}\,.\sin(0+4)\,.\quad.\quad.\quad.\quad.\quad.\quad(\mathrm{D})$$

$$+\frac{1097}{960}k\,e'^{5}\,.\sin(5+0)-\frac{1097}{960}k\,e^{5}\,.\sin(0+5)\quad.\quad.\quad.\quad.\quad.\quad(\mathrm{E})$$

The cosine is

$$\cos(k-k)\,.\cos(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E})$$
$$-\sin(k-k)\,.\sin(\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E})$$

or

$$\cos(k-k)\,.\left\{1-\frac{\mathrm{A}^2+2\,\mathrm{A\,B}+\mathrm{B}^2+2\,\mathrm{A\,C}+2\,\mathrm{A\,D}+2\,\mathrm{B\,C}}{2}+\frac{\mathrm{A}^4+4\,\mathrm{A}^3\mathrm{B}}{24}\right\}$$

$$-\sin(k-k)\,.\left\{\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}+\mathrm{E}-\frac{\mathrm{A}^3+3\,\mathrm{A}^2\,\mathrm{B}+3\,\mathrm{A}^2\,\mathrm{C}+3\,\mathrm{A\,B}^2}{6}+\frac{\mathrm{A}^5}{120}\right\}$$

omitting all products of an order above the fifth.

16. In expanding the powers of A, B, &c., and in multiplying the expansions by $\cos(k-k)$ and $\sin(k-k)$, the rules of (8) must be strictly followed. Thus we find at length for the value of $\cos(k\,v'-k\,v)$:

Principal term,

$$\cos(k-k)$$

Terms of the first order,

$$+k\,e'\,.\cos(\overline{k+1}-k)-k\,e\,.\cos(k-\overline{k-1})$$

Terms of the second order,

$$\left(\tfrac{1}{2}k^2+\tfrac{5}{8}k\right)e'^2\,.\cos\left(\overline{k+2}-k\right)-k^2\,e'\,e\,.\cos\left(\overline{k+1}-\overline{k-1}\right)$$

$$+\left(\tfrac{1}{2}k^2-\tfrac{5}{8}k\right)e^2\,.\cos\left(k-\overline{k-2}\right)$$

Terms of the third order,

$$\left(\tfrac{1}{6}k^3+\tfrac{5}{8}k^2+\tfrac{13}{24}k\right)e'^3.\cos\left(\overline{k+3}-k\right)+\left(-\tfrac{1}{2}k^3-\tfrac{5}{8}k^2\right)e'^2e.\cos\left(\overline{k+2}-\overline{k-1}\right)$$

$$+\left(\frac{1}{2}k^3-\frac{5}{8}k^2\right)e'e^2\,.\cos\left(\overline{k+1}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{6}k^3+\frac{5}{8}k^2-\frac{13}{24}k\right)e^3\,.\cos\left(k-\overline{k-3}\right)$$

Terms of the fourth order,

$$\left(\frac{1}{24}k^4+\frac{5}{16}k^3+\frac{283}{384}k^2+\frac{103}{192}k\right)e'^4\,.\cos\left(\overline{k+4}-k\right)$$

$$+\left(-\frac{1}{6}k^4-\frac{5}{8}k^3-\frac{13}{24}k^2\right)e'^3e\,.\cos\left(\overline{k+3}-\overline{k-1}\right)$$

$$+\left(\frac{1}{4}k^4-\frac{25}{64}k^2\right)e'^2e^2\,.\cos\left(\overline{k+2}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{6}k^4+\frac{5}{8}k^3-\frac{13}{24}k^2\right)e'e^3\,.\cos\left(\overline{k+1}-\overline{k-3}\right)$$

$$+\left(\frac{1}{24}k^4-\frac{5}{16}k^3+\frac{283}{384}k^2-\frac{103}{192}k\right)e^4\,.\cos\left(k-\overline{k-4}\right)$$

Terms of the fifth order,

$$\left(\frac{1}{120}k^5+\frac{5}{48}k^4+\frac{179}{384}k^3+\frac{7}{8}k^2+\frac{1097}{1920}k\right)e'^5\cos\left(\overline{k+5}-k\right)$$

$$+\left(-\frac{1}{24}k^5-\frac{5}{16}k^4-\frac{283}{384}k^3-\frac{103}{192}k^2\right)e'^4e\,.\cos\left(\overline{k+4}-\overline{k-1}\right)$$

$$+\left(\frac{1}{12}k^5+\frac{5}{24}k^4-\frac{23}{192}k^3-\frac{65}{192}k^2\right)e'^3e^2\,.\cos\left(\overline{k+3}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{12}k^5+\frac{5}{24}k^4+\frac{23}{192}k^3-\frac{65}{192}k^2\right)e'^2e^3\,.\cos\left(\overline{k+2}-\overline{k-3}\right)$$

$$+\left(\frac{1}{24}k^5-\frac{5}{16}k^4+\frac{283}{384}k^3-\frac{103}{192}k^2\right)e'e^4\,.\cos\left(\overline{k+1}-\overline{k-4}\right)$$

$$+\left(-\frac{1}{120}k^5+\frac{5}{48}k^4-\frac{179}{384}k^3+\frac{7}{8}k^2-\frac{1097}{1920}k\right)e^5\,.\cos\left(k-\overline{k-5}\right)$$

This development includes every argument whose coefficient is of an order not exceeding the fifth. The coefficients however here exhibited are only the first terms of the series which represent the complete coefficients.

SECTION 4.

Expansion of $-\Gamma_{\frac{1}{2}}^{(k)}$, *to the fifth order.*

17. We suppose $-\frac{m}{\sqrt{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}}}$, the first term in the expression of (14), to be expanded in the form

$$-\tfrac{1}{2}m\,\Gamma_{\frac{1}{2}}^{(0)}-m\Gamma_{\frac{1}{2}}^{(1)}\,.\cos(v'-v)-m\,\Gamma_{\frac{1}{2}}^{(2)}\,.\cos(2\,v'-2\,v)-\&c.$$

$$-m\Gamma_{\frac{1}{2}}^{(k)}\,.\cos(k\,v'-k\,v)-\&c.$$

where $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, &c. are functions of r' and r only.

Let $\frac{1}{\sqrt{\{a'^2-2\,a'\,a\cos(v'-v)+a^2\}}}$

$$=\tfrac{1}{2}\,C_{\frac{1}{2}}^{(0)}+C_{\frac{1}{2}}^{(1)}.\cos(v'-v)+C_{\frac{1}{2}}^{(2)}.\cos(2v'-2v)+\&c.+C_{\frac{1}{2}}^{(k)}.\cos(kv'-kv)+\&c.$$

then $\Gamma_{\frac{1}{2}}^{(k)}$ is the same function of r' and r that $C_{\frac{1}{2}}^{(k)}$ is of a' and a. Consequently, if $r'=a'(1+q')$, $r=a(1+q)$: and if for convenience we use the notation

$$(m,n)\,C_{\frac{1}{2}}^{(k)}$$

to express that which is commonly written

$$a'^m\,.\,a^n\,.\,\frac{d^{m+n}\,C_{\frac{1}{2}}^{(k)}}{d\,a'^m\,.\,d\,a^n}$$

we shall have for $-\Gamma_{\frac{1}{2}}^{(k)}$ the following expression:

$$-C_{\frac{1}{2}}^{(k)}-(1,0)\,C_{\frac{1}{2}}^{(k)}.\,q'-(2,0)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^2}{2}-(3,0)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^3}{6}-(4,0)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^4}{24}-(5,0)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^5}{120}$$

$$-(0,1)\,C_{\frac{1}{2}}^{(k)}.\,q-(1,1)\,C_{\frac{1}{2}}^{(k)}.\,q'q-(2,1)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^2q}{2}-(3,1)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^3q}{6}-(4,1)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^4q}{24}$$

$$-(0,2)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q^2}{2}-(1,2)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'q^2}{2}-(2,2)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^2q^2}{4}-(3,2)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^3q^2}{12}$$

$$-(0,3)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q^3}{6}-(1,3)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'q^3}{6}-(2,3)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'^2q^3}{12}$$

$$-(0,4)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q^4}{24}-(1,4)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q'q^4}{24}$$

$$-(0,5)\,C_{\frac{1}{2}}^{(k)}.\,\frac{q^5}{120}$$

18. The value of r, contracted according to the system of (13), is

$$a\left\{1-e\,.\cos(0+1)-\frac{1}{2}e^2.\cos(0+2)-\frac{3}{8}e^3.\cos(0+3)-\frac{1}{3}e^4.\cos(0+4)\right.$$
$$\left.-\frac{125}{384}e^5.\cos(0+5)\right\}$$

whence $q =$

$$-e\cos(0+1)-\frac{1}{2}e^2.\cos(0+2)-\frac{3}{8}e^3.\cos(0+3)-\frac{1}{3}e^4.\cos(0+4)$$
$$-\frac{125}{384}e^5.\cos(0+5)$$

and a similar expression holds for q'. Substituting these in the expression above, and following strictly the precept of (8), we find for the development of $-\Gamma_{\frac{1}{2}}^{(k)}$,

Principal term,

$$-C_{\frac{1}{2}}^{(k)}$$

Terms of the first order,

$$+(1,0)\,C_{\frac{1}{2}}^{(k)}.e'\cos(1+0)+(0,1)\,C_{\frac{1}{2}}^{(k)}.e\cos(0+1)$$

Terms of the second order,*

$$\left\{\frac{1}{2}(1,0)-\frac{1}{4}(2,0)\right\}C_{\frac{1}{2}}^{(k)}.e'^2\cos(2+0)-\frac{1}{2}(1,1)\,C_{\frac{1}{2}}^{(k)}.e'e\cos(1+1)$$
$$+\left\{\frac{1}{2}(0,1)-\frac{1}{4}(0,2)\right\}C_{\frac{1}{2}}^{(k)}.e^2\cos(0+2)$$

Terms of the third order,

$$\left\{\frac{3}{8}(1,0)-\frac{1}{4}(2,0)+\frac{1}{24}(3,0)\right\}C_{\frac{1}{2}}^{(k)}.e'^3\cos(3+0)$$
$$+\left\{-\frac{1}{4}(1,1)+\frac{1}{8}(2,1)\right\}C_{\frac{1}{2}}^{(k)}.e'^2e\cos(2+1)$$

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* In this and the succeeding expressions, when a cosine is multiplied by the sum of several differential coefficients of $C_{\frac{1}{2}}^{(k)}$, the symbols of differentiation are bracketed together, and $C_{\frac{1}{2}}^{(k)}$ is put at the end of the bracket.

$$+\left\{-\frac{1}{4}(1,1)+\frac{1}{8}(1,2)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'\,e^{2}\cos(1+2)$$

$$+\left\{\frac{3}{8}(0,1)-\frac{1}{4}(0,2)+\frac{1}{24}(0,3)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e^{3}\cos(0+3)$$

Terms of the fourth order,

$$\left\{\frac{1}{3}(1,0)-\frac{1}{4}(2,0)+\frac{1}{16}(3,0)-\frac{1}{192}(4,0)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{4}\cos(4+0)$$

$$+\left\{-\frac{3}{16}(1,1)+\frac{1}{8}(2,1)-\frac{1}{48}(3,1)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{3}\,e\cos(3+1)$$

$$+\left\{-\frac{1}{8}(1,1)+\frac{1}{16}(2,1)+\frac{1}{16}(1,2)-\frac{1}{32}(2,2)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{2}\,e^{2}\cos(2+2)$$

$$+\left\{-\frac{3}{16}(1,1)+\frac{1}{8}(1,2)-\frac{1}{48}(1,3)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'\,e^{3}\cos(1+3)$$

$$+\left\{\frac{1}{3}(0,1)-\frac{1}{4}(0,2)+\frac{1}{16}(0,3)-\frac{1}{192}(0,4)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e^{4}\cos(0+4)$$

Terms of the fifth order,

$$\left\{\frac{125}{384}(1,0)-\frac{25}{96}(2,0)+\frac{5}{64}(3,0)-\frac{1}{96}(4,0)+\frac{1}{1920}(5,0)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{5}\cos(5+0)$$

$$+\left\{-\frac{1}{6}(1,1)+\frac{1}{8}(2,1)-\frac{1}{32}(3,1)+\frac{1}{384}(4,1)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{4}\,e\cos(4+1)$$

$$+\left\{-\frac{3}{32}(1,1)+\frac{1}{16}(2,1)+\frac{3}{64}(1,2)-\frac{1}{96}(3,1)-\frac{1}{32}(2,2)\right.$$
$$\left.+\frac{1}{192}(3,2)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{3}\,e^{2}\cos(3+2)$$

$$+\left\{-\frac{3}{32}(1,1)+\frac{3}{64}(2,1)+\frac{1}{16}(1,2)-\frac{1}{32}(2,2)-\frac{1}{96}(1,3)\right.$$
$$\left.+\frac{1}{192}(2,3)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'^{2}\,e^{3}\cos(2+3)$$

$$+\left\{-\frac{1}{6}(1,1)+\frac{1}{8}(1,2)-\frac{1}{32}(1,3)+\frac{1}{384}(1,4)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e'\,e^{4}\cos(1+4)$$

$$+\left\{\frac{125}{384}(0,1)-\frac{25}{96}(0,2)+\frac{5}{64}(0,3)-\frac{1}{96}(0,4)+\frac{1}{1920}(0,5)\right\}C_{\frac{1}{2}}^{(k)}\,.\,e^{5}\cos(0+5)$$

Every argument is included whose coefficient is of an order not superior to the fifth: but only the lowest order of each coefficient is taken.

SECTION 5.

Selection of the coefficients of cos (13 − 8) *in the development of*

$$-\frac{m}{\sqrt{\{r'^2 - 2r'r \,.\cos(v' - v) + r^2\}}}$$

19. For this purpose, as the general term in the expansion of $-\frac{m}{\sqrt{\{r'^2 - 2r'r \,.\cos(v' - v) + r^2\}}}$ is $- m\,\Gamma_{\frac{1}{2}}^{(k)} .\cos(kv' - kv)$, we ought to multiply together the expressions of (16) and (18), to multiply the product by m, and then giving different values to k to select those terms which have for argument (13 − 8). But without going through this labour we may, when a value is assumed for k, select by the eye the terms required. As we have explained in (7), the values which it is proper to give to k are 8, 9, 10, 11, 12, 13.

20. Thus we obtain the following coefficients of cos (13 − 8):

$$k = 8.$$

$$m \times \left\{ -\frac{239753}{240}(0,0)^{*} + \frac{178109}{768}(1,0) - \frac{4217}{192}(2,0) + \frac{407}{384}(3,0) \right.$$
$$\left. - \frac{5}{192}(4,0) + \frac{1}{3840}(5,0) \right\} C_{\frac{1}{2}}^{(8)} .\, e'^5 \quad \ldots\ldots\ldots \quad (L^{(8)} .\, e'^5)$$

$$k = 9.$$

$$m \times \left\{ \frac{1955097}{384}(0,0) - \frac{88029}{96}(1,0) + \frac{217233}{768}(0,1) + \frac{4041}{64}(2,0) \right.$$
$$- \frac{9781}{192}(1,1) - \frac{189}{96}(3,0) + \frac{449}{128}(2,1) + \frac{3}{128}(4,0) - \frac{7}{64}(3,1)$$
$$\left. + \frac{1}{768}(4,1) \right\} C_{\frac{1}{2}}^{(9)} .\, e'^4 e \quad \ldots\ldots\ldots \quad (L^{(9)} .\, e'^4 e)$$

---

* By $(0,0)\, C_{\frac{1}{2}}^{(8)}$ is meant the same as $C_{\frac{1}{2}}^{(8)}$.

$k = 10.$

$$\begin{aligned} m \times \Big\{ & - \frac{492625}{48}(0,0) + \frac{86275}{64}(1,0) - \frac{53485}{48}(0,1) - \frac{1925}{32}(2,0) \\ & + \frac{9367}{64}(1,1) - \frac{2815}{96}(0,2) + \frac{175}{192}(3,0) - \frac{209}{32}(2,1) + \frac{493}{128}(1,2) \\ & + \frac{19}{192}(3,1) - \frac{11}{64}(2,2) + \frac{1}{384}(3,2) \Big\} C_{\frac{1}{2}}^{(10)} . e'^3 e^2 \quad . \; . \; . \quad (L^{(10)} . e'^3 e^2) \end{aligned}$$

$k = 11.$

$$\begin{aligned} m \times \Big\{ & \frac{492107}{48}(0,0) - \frac{20999}{24}(1,0) + \frac{52283}{32}(0,1) + \frac{913}{48}(2,0) \\ & - \frac{2231}{16}(1,1) + \frac{2695}{32}(0,2) + \frac{97}{32}(2,1) - \frac{115}{16}(1,2) + \frac{539}{384}(0,3) \\ & + \frac{5}{32}(2,2) - \frac{23}{192}(1,3) + \frac{1}{384}(2,3) \Big\} C_{\frac{1}{2}}^{(11)} . e'^2 e^3 \quad . \; . \; . \quad (L^{(11)} . e'^2 e^3) \end{aligned}$$

$k = 12.$

$$\begin{aligned} m \times \Big\{ & - \frac{20337}{4}(0,0) + \frac{6779}{32}(1,0) - \frac{2117}{2}(0,1) + \frac{2119}{48}(1,1) \\ & - \frac{321}{4}(0,2) + \frac{107}{32}(1,2) - \frac{21}{8}(0,3) + \frac{7}{64}(1,3) - \frac{1}{32}(0,4) \\ & + \frac{1}{768}(1,4) \Big\} C_{\frac{1}{2}}^{(12)} . e' e^4 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (L^{(12)} . e' e^4) \end{aligned}$$

$k = 13.$

$$\begin{aligned} m \times \Big\{ & \frac{240643}{240}(0,0) + \frac{24571}{96}(0,1) + \frac{1219}{48}(0,2) + \frac{235}{192}(0,3) \\ & + \frac{11}{384}(0,4) + \frac{1}{3840}(0,5) \Big\} C_{\frac{1}{2}}^{(13)} . e^5 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (L^{(13)} . e^5) \end{aligned}$$

The arguments of the cosines multiplied respectively by these coefficients, it must be recollected, are not similar. Their form will be determined by the considerations mentioned in (10).

21. The next term of R to be developed, by (14), is

$$-m\,.\,\frac{r'r}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{3}{2}}}\,.\,f^2\,.\cos(v'+v-2\theta)$$

We shall put $\Gamma_{\frac{3}{2}}^{(k)}$ for the general term in the expansion

$$\frac{r'r}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{3}{2}}}=\tfrac{1}{2}\Gamma_{\frac{3}{2}}^{(0)}+\Gamma_{\frac{3}{2}}^{(1)}.\cos(v'-v)+\Gamma_{\frac{3}{2}}^{(2)}.\cos(2v'-2v)+\&c.;$$

And $C_{\frac{3}{2}}^{(k)}$ for the general term in the expansion

$$\frac{a'a}{\{a'^2-2a'a\,.\cos(v'-v)+a^2\}^{\frac{3}{2}}}=\tfrac{1}{2}C_{\frac{3}{2}}^{(0)}+C_{\frac{3}{2}}^{(1)}\cos(v'-v)+C_{\frac{3}{2}}^{(2)}\cos(2v'-2v)+\&c.$$

SECTION 6.

Development of $f^2\,.\cos(v'+v-2\theta)$, *to the fifth order.*

22. As the multiplier f^2 is of the second order, we want $\cos(v'+v-2\theta)$ only to the third order. Now, by (13), $v'+v-2\theta=$

$$(1+1)-2\theta$$

$$+\,2e'\sin(1+0)+2e\,.\sin(0+1)\,.\,.\,.\,.\,.\,.\,.\,.\,.\,.\quad\text{(A)}$$

$$+\,\frac{5}{4}e'^2.\sin(2+0)+\frac{5}{4}e^2.\sin(0+2)\,.\,.\,.\,.\,.\,.\,.\,.\,.\quad\text{(B)}$$

$$+\,\frac{13}{12}e'^3\sin(3+0)+\frac{13}{12}e^3.\sin(0+3)\,.\,.\,.\,.\,.\,.\,.\,.\,.\quad\text{(C)}$$

Its cosine, as in (15), is

$$\cos(1+1-2\theta)\,.\left\{1-\frac{A^2+2AB}{2}\right\}-\sin(1+1-2\theta)\,.\left\{A+B+C-\frac{A^3}{6}\right\}$$

Following the rule of (8) in the expansion, we find for the value of cos $(v'+v-2\theta)$.

Principal Term,

$$\cos(1+1-2\theta)$$

Terms of the first order,

$$+ e' . \cos(2 + 1 - 2\theta) + e . \cos(1 + 2 - 2\theta)$$

Terms of the second order,

$$+ \frac{9}{8} e'^2 . \cos(3 + 1 - 2\theta) + e' e . \cos(2 + 2 - 2\theta) + \frac{9}{8} e^2 . \cos(1 + 3 - 2\theta)$$

Terms of the third order,

$$+ \frac{4}{3} e'^3 . \cos(4 + 1 - 2\theta) + \frac{9}{8} e'^2 e . \cos(3 + 2 - 2\theta)$$

$$+ \frac{9}{8} e' e^2 . \cos(2 + 3 - 2\theta) + \frac{4}{3} e^3 . \cos(1 + 4 - 2\theta)$$

On multiplying this by f^2 it will readily be seen that f^2 in the coefficient is always accompanied by -2θ in the argument, and that there is a necessary connexion between them. We may therefore omit 2θ; and thus we have for the development of $f^2 . \cos(v' + v - 2\theta)$

Term of the second order,

$$f^2 . \cos(1 + 1).$$

Terms of the third order,

$$+ e' f^2 . \cos(2 + 1) + e f^2 . \cos(1 + 2).$$

Terms of the fourth order,

$$+ \frac{9}{8} e'^2 f^2 . \cos(3 + 1) + e' e f^2 . \cos(2 + 2) + \frac{9}{8} e^2 f^2 . \cos(1 + 3)$$

Terms of the fifth order,

$$+ \frac{4}{3} e'^3 f^2 . \cos(4 + 1) + \frac{9}{8} e'^2 e f^2 . \cos(3 + 2) + \frac{9}{8} e' e^2 f^2 . \cos(2 + 3)$$

$$+ \frac{4}{3} e^3 f^2 . \cos(1 + 4)$$

SECTION 7.

Development of $\cos (kv' - kv) \,.\, f^2 \,.\, \cos (v' + v - 2\theta)$*, to the fifth order.*

23. We must multiply the expression in (16), (of which only the terms to the third order will be wanted), by the expression just formed, according to the rule of (8). Thus we obtain the following expression:

Term of the second order,

$$\frac{1}{2} f^2 \,.\, \cos (\overline{k+1} - \overline{k-1}).$$

Terms of the third order,

$$\left(\frac{1}{2} k + \frac{1}{2}\right) e' f^2 \,.\, \cos (\overline{k+2} - \overline{k-1}) + \left(-\frac{1}{2} k + \frac{1}{2}\right) e f^2 \cos (\overline{k+1} - \overline{k-2});$$

Terms of the fourth order,

$$\left(\frac{1}{4} k^2 + \frac{13}{16} k + \frac{9}{16}\right) e'^2 f^2 \,.\, \cos (\overline{k+3} - \overline{k-1})$$

$$+ \left(-\frac{1}{2} k^2 + \frac{1}{2}\right) e' e f^2 \,.\, \cos (\overline{k+2} - \overline{k-2})$$

$$+ \left(\frac{1}{4} k^2 - \frac{13}{16} k + \frac{9}{16}\right) e^2 f^2 \,.\, \cos (\overline{k+1} - \overline{k-3})$$

Terms of the fifth order,

$$\left(\frac{1}{12} k^3 + \frac{9}{16} k^2 + \frac{55}{48} k + \frac{2}{3}\right) e'^3 f^2 \,.\, \cos (\overline{k+4} - \overline{k-1})$$

$$+ \left(-\frac{1}{4} k^3 - \frac{9}{16} k^2 + \frac{1}{4} k + \frac{9}{16}\right) e'^2 e f^2 \,.\, \cos (\overline{k+3} - \overline{k-2})$$

$$+ \left(\frac{1}{4} k^3 - \frac{9}{16} k^2 - \frac{1}{4} k + \frac{9}{16}\right) e' e^2 f^2 \,.\, \cos (\overline{k+2} - \overline{k-3})$$

$$+ \left(-\frac{1}{12} k^3 + \frac{9}{16} k^2 - \frac{55}{48} k + \frac{2}{3}\right) e^3 f^2 \,.\, \cos (\overline{k+1} - \overline{k-4})$$

SECTION 8.

Selection of the coefficients of cos (13 — 8) *in the development of*

$$-m \, . \, \frac{r' r}{\{r'^2 - 2 r' r \, . \cos(v' - v) + r^2\}^{\frac{3}{2}}} f^2 \, . \cos(v' + v - 2\,\theta).$$

24. The general term of the expansion is $-m \, . \, \Gamma^{(k)}_{\frac{3}{2}} \, . \cos(k v' - k v) \, . f^2 \, . \cos(v' + v - 2\,\theta)$. The expression for $\cos(k v' - k v) \, . f^2 \, . \cos(v' + v - 2\,\theta)$ we have just found; and the expression for $-\Gamma^{(k)}_{\frac{3}{2}}$ will be in all respects similar to that for $-\Gamma^{(k)}_{\frac{1}{2}}$ in (18), putting $C^{(k)}_{\frac{3}{2}}$ for $C^{(k)}_{\frac{1}{2}}$. Observing that k cannot be less than 9 or greater than 12, and selecting for the different values of k the terms whose combination produces (13 — 8), we get the following coefficients:

$k = 9.$

$$m \times \left\{ -\frac{2815}{24}(0,0) + \frac{493}{32}(1,0) - \frac{11}{16}(2,0) + \frac{1}{96}(3,0) \right\} C^{(9)}_{\frac{3}{2}} \, . \, e'^3 f^2 \, . \, . \, (M^{(9)} e'^3 f^2)$$

$k = 10.$

$$m \times \left\{ \frac{4851}{16}(0,0) - \frac{207}{8}(1,0) + \frac{539}{32}(0,1) + \frac{9}{16}(2,0) - \frac{23}{16}(1,1) \right.$$
$$\left. + \frac{1}{32}(2,1) \right\} C^{(10)}_{\frac{3}{2}} \, . \, e'^2 e f^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (M^{(10)} \, . \, e'^2 e f^2)$$

$k = 11.$

$$m \times \left\{ -\frac{525}{2}(0,0) + \frac{175}{16}(1,0) - \frac{57}{2}(0,1) + \frac{19}{16}(1,1) - \frac{3}{4}(0,2) \right.$$
$$\left. + \frac{1}{32}(1,2) \right\} C^{(11)}_{\frac{3}{2}} \, . \, e' e^2 f^2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (M^{(11)} \, . \, e' e^2 f^2)$$

$k = 12.$

$$m \times \left\{ \frac{913}{12}(0,0) + \frac{97}{8}(0,1) + \frac{5}{8}(0,2) + \frac{1}{96}(0,3) \right\} C^{(12)}_{\frac{3}{2}} \, . \, e^3 f^2 \; . \; . \; (M^{(12)} \, . \, e^3 f^2)$$

The arguments of the cosines multiplied by these coefficients are not similar; their forms may be found by the reasoning in (10).

25. The next term of R to be developed, by (14), is

$$-m\,.\,\frac{3}{4}\,\frac{r'^2 r^2}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}f^4\,.\cos(2v'+2v-4\theta).$$

We shall put $\Gamma_{\frac{5}{2}}^{(k)}$ for the general term in the expansion

$$\frac{r'^2 r^2}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}=\tfrac{1}{2}\Gamma_{\frac{5}{2}}^{(0)}+\Gamma_{\frac{5}{2}}^{(1)}\,.\cos(v'-v)+\Gamma_{\frac{5}{2}}^{(2)}\,.\cos(2v'-2v)+\&c.$$

and $C_{\frac{5}{2}}^{(k)}$ for the general term in the expansion

$$\frac{a'^2 a^2}{\{a'^2-2a'a\,.\cos(v'-v)+a^2\}^{\frac{5}{2}}}=\tfrac{1}{2}C_{\frac{5}{2}}^{(0)}+C_{\frac{5}{2}}^{(1)}\,.\cos(v'-v)+C_{\frac{5}{2}}^{(2)}\,.\cos(2v'-2v)+\&c.$$

SECTION 9.

Development of $\cos(kv'-kv)\,.f^4\,.\cos(2v'+2v-4\theta)$, *to the fifth order.*

26. As the multiplier f^4 is of the fourth order, we need to develope $\cos(2v'+2v-4\theta)$ only to the first order. Now by (13), $2v'+2v-4\theta=$

$$(2+2)-4\theta$$
$$+4e'\,.\sin(1+0)+4e\,.\sin(0+1)$$

and consequently $\cos(2v'+2v-4\theta)=$

$$\cos(2+2-4\theta)-\sin(2+2-4\theta)\,.\{4e'\,.\sin(1+0)+4e\,.\sin(0+1)\}$$
$$=\cos(2+2-4\theta)$$
$$+2e'\cos(3+2-4\theta)+2e\,.\cos(2+3-4\theta)$$

Multiplying this by f^4 it will be seen, as in (22), that we may omit 4θ in the argument. Thus we have for the development of $f^4\,.\cos(2v'+2v-4\theta)$,

Term of the fourth order,

$$f^4\,.\cos(2+2).$$

Terms of the fifth order,

$$+ 2 e' f^4 . \cos (3 + 2) + 2 e f^4 . \cos (2 + 3).$$

27. This is now to be multiplied by $\cos (k v' - k v)$, the expansion of which has been performed in (16). Effecting this operation, we have for the development of $\cos (k v' - k v) . f^4 . \cos (2 v' + 2 v - 4 \theta)$,

Term of the fourth order,

$$\frac{1}{2} f^4 . \cos (\overline{k + 2} - \overline{k - 2})$$

Terms of the fifth order,

$$\left(\frac{1}{2} k + 1\right) e' f^4 . \cos (\overline{k + 3} - \overline{k - 2}) + \left(- \frac{1}{2} k + 1\right) e f^4 . \cos (\overline{k + 2} - \overline{k - 3})$$

SECTION 10.

Selection of the coefficients of cos (13 − 8) *in the development of*

$$- m . \frac{3}{4} . \frac{r'^2 r^2}{\{r'^2 - 2 r' r . \cos (v' - v) + r^2\}^{\frac{5}{2}}} . f^4 . \cos (2 v' + 2 v - 4 \theta).$$

28. We must suppose the expression of (27) to be multiplied by $\frac{3}{4} m$, and by the expression for $- \Gamma_{\frac{5}{2}}^{(k)}$ (which will be formed from that of (18), putting $C_{\frac{5}{2}}^{(k)}$ for $C_{\frac{3}{2}}^{(k)}$). Then giving to k different values, we must select the terms in the product whose argument is (13 − 8). It is easily seen that 10 and 11 are the only admissible values of k. Thus we get these coefficients;

$$k = 10.$$

$$m \times \left\{ - \frac{9}{2} (0,0) + \frac{3}{16} (1,0) \right\} C_{\frac{5}{2}}^{(10)} . e' f^4 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (N^{(10)} . e' f^4)$$

$$k = 11.$$

$$m \times \left\{ \frac{27}{8} (0,0) + \frac{3}{16} (0,1) \right\} C_{\frac{5}{2}}^{(11)} . e f^4 \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (N^{(11)} . e f^4)$$

29. The terms collected in (20), (24), and (28), form the complete coefficient of cos (13 − 8) in the development of R to the fifth order. The arguments of the cosines multiplied by the different series are all different; so that there are twelve different terms to be calculated. Using the symbols $\overset{(8)}{\mathrm{L}}$, &c., the complete term is expressed thus:

$$
\begin{aligned}
& \overset{(8)}{\mathrm{L}} \,.\, e'^5 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 5\,\varpi'\} \\
+\; & \overset{(9)}{\mathrm{L}} \,.\, e'^4\, e \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 4\,\varpi' - \varpi\} \\
+\; & \overset{(10)}{\mathrm{L}} \,.\, e'^3\, e^2 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 3\,\varpi' - 2\,\varpi\} \\
+\; & \overset{(11)}{\mathrm{L}} \,.\, e'^2\, e^3 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 2\,\varpi' - 3\,\varpi\} \\
+\; & \overset{(12)}{\mathrm{L}} \,.\, e'\, e^4 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - \varpi' - 4\,\varpi\} \\
+\; & \overset{(13)}{\mathrm{L}} \,.\, e^5 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 5\,\varpi\} \\
+\; & \overset{(9)}{\mathrm{M}} \,.\, e'^3 f^2 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 3\,\varpi' - 2\,\theta\} \\
+\; & \overset{(10)}{\mathrm{M}} \,.\, e'^2\, e f^2 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 2\,\varpi' - \varpi - 2\,\theta\} \\
+\; & \overset{(11)}{\mathrm{M}} \,.\, e'\, e^2 f^2 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - \varpi' - 2\,\varpi - 2\,\theta\} \\
+\; & \overset{(12)}{\mathrm{M}} \,.\, e^3 f^2 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - 3\,\varpi - 2\,\theta\} \\
+\; & \overset{(10)}{\mathrm{N}} \,.\, e' f^4 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - \varpi' - 4\,\theta\} \\
+\; & \overset{(11)}{\mathrm{N}} \,.\, e f^4 \,.\, \cos\{13\,(n't+\epsilon') - 8\,(nt+\epsilon) - \varpi - 4\,\theta\}
\end{aligned}
$$

SECTION 11.

Considerations on the numerical calculation of the inequalities in the Earth's motion depending on this term.

30. If we examine the expressions of (2), it will appear that the values of all may be deduced with little trouble from the terms above, except that depending on $\frac{d\,\mathrm{R}}{d\,a'}$. Since a' enters only into the coefficients, $\frac{d\,\mathrm{R}}{d\,a'}$ will be produced by

differentiating the coefficients and retaining the same cosines. The coefficients will be differentiated by changing $(0,0)\ C_{\frac{1}{2}}^{(8)}$ into $\frac{1}{a'}(1,0)\ C_{\frac{1}{2}}^{(8)}$, $(3,2)\ C_{\frac{1}{2}}^{(8)}$ into $\frac{1}{a'}(4,2)\ C_{\frac{1}{2}}^{(8)} + \frac{3}{a'}(3,2)\ C_{\frac{1}{2}}^{(8)}$, &c. Thus new terms will be introduced whose calculation is rather troublesome. It is desirable, then, to inquire whether it is probable that the term depending on $\frac{d\,\mathrm{R}}{d\,a'}$ will be comparable in magnitude to the other term which has the same argument.

31. Now if we put $\mathrm{A} \,.\, \cos\{13\,(n'\,t + \epsilon') - 8\,(n\,t + \epsilon) + \mathrm{B}\}$ or $\mathrm{A}\cos(13 - 8)$, for one of the terms, we find

$$\frac{d\,n'}{d\,t} = -\,3\,.\,13\,.\,\frac{n'^2\,a'}{\mu'}\,.\,\mathrm{A}\,.\,\sin(13-8)$$

whence

$$n' = \mathrm{N}' + \frac{3\,.\,13\,.\,n'^2\,a'}{(13\,n' - 8\,n)\,\mu'}\,\mathrm{A}\,.\,\cos(13-8)$$

(where N′ is constant and = mean value of n')

$$\frac{d\,\epsilon'}{d\,t} = +\,3\,.\,13\,.\,\frac{n'^2\,a'}{\mu'}\,\mathrm{A}\,.\,t\,.\,\sin(13-8) + \frac{2\,n'\,a'^2}{\mu'}\,.\,\frac{d\,\mathrm{A}}{d\,a'}\cos(13-8)$$

whence

$$\epsilon' = \mathrm{E}' - \frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)\,\mu'}\,\mathrm{A}\,.\,t\,.\,\cos(13-8) + \frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)^2\,\mu'}\,\mathrm{A}\,.\,\sin(13-8)$$
$$+ \frac{2\,n'\,a'^2}{(13\,n' - 8\,n)\,\mu'}\,.\,\frac{d\,\mathrm{A}}{d\,a'}\sin(13-8)$$

(where E′ is constant and = mean value of ϵ')

and $n'\,t + \epsilon'$ (which, by (1), is the first term of v') becomes

$$\mathrm{N}'\,t + \mathrm{E}' + \left\{\frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)^2\,\mu'}\,\mathrm{A} + \frac{2\,n'\,a'^2}{(13\,n' - 8\,n)\,\mu'}\,.\,\frac{d\,\mathrm{A}}{d\,a'}\right\}\sin(13-8).$$

The ratio of the two coefficients of the inequality sin (13 − 8) is

$$\frac{39}{2}\,.\,\frac{n'}{13\,n' - 8\,n}\,\mathrm{A} : a'\,\frac{d\,\mathrm{A}}{d\,a'}$$

or nearly $\quad 4800 \times \mathrm{A} : a'\,\frac{d\,\mathrm{A}}{d\,a'}.$

It will be seen hereafter, that for any one of the terms whose union composes $\mathrm{L}^{(8)}$, &c., $a'\frac{d\,\mathrm{A}}{d\,a'}$ is greater than $-\mathrm{A}$, and that it may, on the mean of values, be said to differ little from $-12\,\mathrm{A}$. This reduces the ratio of the terms to 400 : 1. Now though we cannot assert that the sum of one set of terms will have to the sum of the other set of terms a ratio at all similar to this, yet the great disproportion of the terms related to each other seems sufficiently to justify us in the *à priori* assertion that the terms depending on $\frac{d\,\mathrm{R}}{d\,a'}$ are not worth calculating. It will readily be seen that the terms depending on $\frac{d\,\mathrm{R}}{d\,e'}$ are still more insignificant than those depending on $\frac{d\,\mathrm{R}}{d\,a'}$.

32. We stated in (1) that the variations of the elements would be sufficiently taken into account in the expression for R if we put $\mathrm{E}+\mathrm{F}\,t$ for e, &c.; which amounts to taking only the secular variations. There will be no difficulty in doing this for e', e, ϖ', ϖ, f, and θ: but if such terms existed in the approximate expressions for a' and a, they would require the use of the differentials $\frac{d\,\mathrm{R}}{d\,a'}$, $\frac{d\,\mathrm{R}}{d\,a}$. But a' and a have no secular variations: and therefore these differentials are not wanted. We may therefore proceed at once with the numerical calculation of the terms $\mathrm{L}^{(8)}$, $\mathrm{L}^{(9)}$, &c.

SECTION 12.

Numerical calculation of $\mathrm{C}_{\frac{1}{2}}^{(0)}$, $\mathrm{C}_{\frac{1}{2}}^{(1)}$, $\mathrm{C}_{\frac{1}{2}}^{(2)}$, *&c.*, $\mathrm{C}_{\frac{3}{2}}^{(k)}$, $\mathrm{C}_{\frac{5}{2}}^{(k)}$, *&c. to* $\mathrm{C}_{\frac{11}{2}}^{(k)}$.

33. If we put $\pi-2\,\omega$ for $v'-v$, we have

$$\frac{1}{\sqrt{\{a'^2+2\,a'a\,.\cos 2\,\omega+a^2\}}}=\tfrac{1}{2}\,\mathrm{C}_{\frac{1}{2}}^{(0)}-\mathrm{C}_{\frac{1}{2}}^{(1)}\,.\cos 2\,\omega+\mathrm{C}_{\frac{1}{2}}^{(2)}\,.\cos 4\,\omega-\&\text{c}.$$

Integrating both sides with respect to ω, from $\omega=0$ to $\omega=\frac{\pi}{2}$, and putting S_ω for the symbol of integration with respect to ω between these limits,

$$\mathrm{S}_\omega\,.\,\frac{1}{\sqrt{\{a'^2+2\,a'\,a\,.\cos 2\,\omega+a^2\}}}=\frac{\pi}{4}\,\mathrm{C}_{\frac{1}{2}}^{(0)}$$

whence

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi} \, S_{\omega} \, . \, \frac{1}{\sqrt{\{a'^2 + 2a'a \, . \cos 2\omega + a^2\}}}$$

or, putting α for $\frac{a}{a'}$,

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi a'} \, . \, S_{\omega} \, \frac{1}{\sqrt{\{1 + 2\alpha \cos 2\omega + \alpha^2\}}}.$$

Now let $\sin \omega' = \frac{\sin 2\omega}{\sqrt{\{1 + 2\alpha \cos 2\omega + \alpha^2\}}}$; and $\alpha' = \frac{1 - \sqrt{1 - \alpha^2}}{1 + \sqrt{1 - \alpha^2}}$: after substitution it is found that

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi a'} (1 + \alpha') \, . \, S_{\omega'} \, . \, \frac{1}{\sqrt{\{1 + 2\alpha' \cos 2\omega' + \alpha'^2\}}}$$

In the same manner, making $\sin \omega'' = \frac{\sin 2\omega'}{\sqrt{\{1 + 2\alpha' \cos 2\omega' + \alpha'^2\}}}$: $\alpha'' = \frac{1 - \sqrt{1 - \alpha'^2}}{1 + \sqrt{1 - \alpha'^2}}$: and so on, we get for $C_{\frac{1}{2}}^{(0)}$ the expression

$$\frac{4}{\pi a'} (1 + \alpha') \, (1 + \alpha'') \ldots\ldots (1 + \alpha^{(n)}) \, . \, S_{\omega^{(n)}} \, \frac{1}{\sqrt{\{1 + 2\alpha^{(n)} \cos 2\omega^{(n)} + \alpha^{(n)2}\}}}$$

The values of α', α'', &c. decrease very rapidly; and when $\alpha^{(n)}$ is insensible, $S_{\omega^{(n)}} \frac{1}{\sqrt{\{1 + 2\alpha^{(n)} \cos 2\omega^{(n)} + \alpha^{(n)2}\}}}$ becomes $S_{\omega^{(n)}} \, . \, 1$ or $\frac{\pi}{2}$. Consequently

$$C_{\frac{1}{2}}^{(0)} = \frac{2}{a'} (1 + \alpha') \, (1 + \alpha'') \, (1 + \alpha''') \, . \, \text{\&c.}$$

the factors being continued till $\alpha^{(n)}$ becomes insensible. The calculation is very easy; for, if we make $\sin \beta = \alpha$, $\sin \beta' = \tan^2 \frac{\beta}{2}$, $\sin \beta'' = \tan^2 \frac{\beta'}{2}$, &c. then $C_{\frac{1}{2}}^{(0)} = \frac{2}{a'} \sec^2 \frac{\beta}{2} \, . \sec^2 \frac{\beta'}{2} \, . \sec^2 \frac{\beta''}{2} \, .$ &c. For Venus and the Earth (Méc. Cél. liv. VI.) α or $\frac{a}{a'} = 0{,}7233323$: using this number in the calculation, $C_{\frac{1}{2}}^{(0)} = \frac{1}{a'} \times 2{,}386375$.

34. Again, $\frac{\cos 2\omega}{\sqrt{\{a'^2 + 2a'a \, . \cos 2\omega + a^2\}}} = \frac{1}{2} \, C_{\frac{1}{2}}^{(0)} \, . \cos 2\omega - \frac{1}{2} \, C_{\frac{1}{2}}^{(1)} (1 + \cos 4\omega) + \frac{1}{2} \, C_{\frac{1}{2}}^{(2)} (\cos 2\omega + \cos 6\omega) - \text{\&c.}$; integrating between the same limits as before,

$$C_{\frac{1}{2}}^{(1)} = -\frac{4}{\pi a'} S_{\omega} . \frac{\cos 2\omega}{\sqrt{\{1 + 2\alpha \cos 2\omega + \alpha^2\}}}$$

Making the same substitution as in (33), there will be produced three terms; of which one vanishes in the definite integral, the second is similar to the expression of this article, and the third similar to that of (33). Making a similar substitution in the second term, new terms are produced. Pursuing this method, it will be found that the only terms whose values are ultimately sensible are those which are similar to the expression of (33): and at last we get

$$C_{\frac{1}{2}}^{(1)} = C_{\frac{1}{2}}^{(0)} . \left\{\frac{\sin\beta}{2} + \frac{\sin\beta}{2} . \frac{\sin\beta'}{2} + \frac{\sin\beta}{2} . \frac{\sin\beta'}{2} . \frac{\sin\beta''}{2} + \&c.\right\} = \frac{1}{a'} \times 0{,}9424137$$

35. Putting χ for $v' - v$, and differentiating with respect to χ the logarithms of both sides of the equation

$$\frac{1}{\sqrt{\{a'^2 - 2a'a . \cos\chi + a^2\}}} = \frac{1}{2} C_{\frac{1}{2}}^{(0)} + C_{\frac{1}{2}}^{(1)} . \cos\chi + C_{\frac{1}{2}}^{(2)} \cos 2\chi + \&c.$$

multiplying out the denominators, and comparing the coefficients of $\cos k\chi$,

$$C_{\frac{1}{2}}^{(k+1)} = \frac{2k}{2k+1}\left(\frac{1}{\alpha} + \alpha\right) C_{\frac{1}{2}}^{(k)} - \frac{2k-1}{2k+1} C_{\frac{1}{2}}^{(k-1)}$$

where $\frac{1}{\alpha} + \alpha = 2{,}1058226$. Making k successively 1, 2, 3, 4, &c., we get the following values:

$$\begin{array}{lll}
C_{\frac{1}{2}}^{(0)} = \frac{1}{a'} \times 2{,}3863750^{*} & C_{\frac{1}{2}}^{(6)} = \frac{1}{a'} \times 0{,}0903724 & C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0{,}0093812 \\
C_{\frac{1}{2}}^{(1)} = \frac{1}{a'} \times 0{,}9424137 & C_{\frac{1}{2}}^{(7)} = \frac{1}{a'} \times 0{,}0609432 & C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0{,}0065274 \\
C_{\frac{1}{2}}^{(2)} = \frac{1}{a'} \times 0{,}5275791 & C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 0{,}0414571 & C_{\frac{1}{2}}^{(14)} = \frac{1}{a'} \times 0{,}0045503 \\
C_{\frac{1}{2}}^{(3)} = \frac{1}{a'} \times 0{,}3233422 & C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0{,}0283925 & C_{\frac{1}{2}}^{(15)} = \frac{1}{a'} \times 0{,}0031744 \\
C_{\frac{1}{2}}^{(4)} = \frac{1}{a'} \times 0{,}2067875 & C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0{,}0195495 & C_{\frac{1}{2}}^{(16)} = \frac{1}{a'} \times 0{,}0022123 \\
C_{\frac{1}{2}}^{(5)} = \frac{1}{a'} \times 0{,}1355852 & C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0{,}0135189 & C_{\frac{1}{2}}^{(17)} = \frac{1}{a'} \times 0{,}0015356 \\
 & & C_{\frac{1}{2}}^{(18)} = \frac{1}{a'} \times 0{,}0010554
\end{array}$$

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* LAPLACE's numbers, which are somewhat different from these, are computed by the less accurate method of summing a slowly converging series.

36. For the calculation of the terms $C_{\frac{3}{2}}^{(k)}$, $C_{\frac{5}{2}}^{(k)}$, &c., we shall adopt the general notation

$$\frac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s}}=\frac{1}{2}C_s^{(0)}+C_s^{(1)}\cos\chi+C_s^{(2)}\cos 2\chi+\&c.$$

which, it will be seen, includes those of (17), (21), and (25); and proceeding as in (35) we shall find this general equation

$$C_s^{(k+1)}=\frac{k}{k+1-s}\left(\frac{1}{\alpha}+\alpha\right)C_s^{(k)}-\frac{k-1+s}{k+1-s}C_s^{(k-1)}$$

And since $\dfrac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s}}=$

$$\left(\frac{1}{\alpha}+\alpha-2\cos\chi\right)\times\frac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s+1}}$$

we find on substituting the expansions and comparing the coefficients of $\cos k\chi$,

$$C_s^{(k)}=\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k)}-C_{s+1}^{(k-1)}-C_{s+1}^{(k+1)}$$

Removing $C_{s+1}^{(k+1)}$ by means of the relation just found (putting $s+1$ for s)

$$C_s^{(k)}=-\frac{s}{k-s}\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k)}+\frac{2s}{k-s}C_{s+1}^{(k-1)}$$

In nearly the same manner,

$$C_s^{(k-1)}=\frac{s}{k+s-1}\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k-1)}-\frac{2s}{k+s-1}C_{s+1}^{(k)}$$

Eliminating $C_{s+1}^{(k-1)}$,

$$C_{s+1}^{(k)}=\frac{2(k+s-1)}{s}\cdot\frac{1}{\left(\frac{1}{\alpha}-\alpha\right)^2}C_s^{(k-1)}-\frac{k-s}{s}\cdot\frac{\frac{1}{\alpha}+\alpha}{\left(\frac{1}{\alpha}-\alpha\right)^2}C_s^{(k)}$$

If in this we substitute the value of $C_s^{(k)}$ in terms of $C_s^{(k-1)}$ and $C_s^{(k+1)}$, given by the relation above,

$$C_{s+1}^{(k)} = \frac{1}{\left(\frac{1}{\alpha} - \alpha\right)^2} \left\{ \frac{(k+s)(k+s-1)}{ks} C_s^{(k-1)} - \frac{(k-s)(k-s+1)}{ks} C_s^{(k+1)} \right\}$$

in which $\frac{1}{\left(\frac{1}{\alpha} - \alpha\right)^2}$ or $\frac{\alpha^2}{(1-\alpha^2)^2}$ is $\frac{\sin^2\beta}{\cos^4\beta} = 2{,}3015505$

37. Making $s = \frac{1}{2}$, $C_{\frac{3}{2}}^{(k)} = \frac{\sin^2\beta}{2\cos^4\beta} \cdot \frac{4k^2-1}{k} \cdot \left(C_{\frac{1}{2}}^{(k-1)} - C_{\frac{1}{2}}^{(k+1)}\right)$. Using this formula,

$$C_{\frac{3}{2}}^{(4)} = \frac{1}{a'} \times 3{,}403041 \qquad C_{\frac{3}{2}}^{(9)} = \frac{1}{a'} \times 0{,}904785 \qquad C_{\frac{3}{2}}^{(14)} = \frac{1}{a'} \times 0{,}215803$$

$$C_{\frac{3}{2}}^{(5)} = \frac{1}{a'} \times 2{,}652559 \qquad C_{\frac{3}{2}}^{(10)} = \frac{1}{a'} \times 0{,}682935 \qquad C_{\frac{3}{2}}^{(15)} = \frac{1}{a'} \times 0{,}161251$$

$$C_{\frac{3}{2}}^{(6)} = \frac{1}{a'} \times 2{,}047192 \qquad C_{\frac{3}{2}}^{(11)} = \frac{1}{a'} \times 0{,}513799 \qquad C_{\frac{3}{2}}^{(16)} = \frac{1}{a'} \times 0{,}120579$$

$$C_{\frac{3}{2}}^{(7)} = \frac{1}{a'} \times 1{,}568093 \qquad C_{\frac{3}{2}}^{(12)} = \frac{1}{a'} \times 0{,}385521 \qquad C_{\frac{3}{2}}^{(17)} = \frac{1}{a'} \times 0{,}090452$$

$$C_{\frac{3}{2}}^{(8)} = \frac{1}{a'} \times 1{,}193991 \qquad C_{\frac{3}{2}}^{(13)} = \frac{1}{a'} \times 0{,}288655$$

38. Making $s = \frac{3}{2}$, $C_{\frac{5}{2}}^{(k)} = \frac{\sin^2\beta}{6\cos^4\beta} \cdot \left\{ \frac{(2k+3)(2k+1)}{k} C_{\frac{3}{2}}^{(k-1)} - \frac{(2k-3)(2k-1)}{k} C_{\frac{3}{2}}^{(k+1)} \right\}$. By the use of this formula,

$$C_{\frac{5}{2}}^{(5)} = \frac{1}{a'} \times 27{,}43922 \qquad C_{\frac{5}{2}}^{(9)} = \frac{1}{a'} \times 12{,}88246 \qquad C_{\frac{5}{2}}^{(13)} = \frac{1}{a'} \times 5{,}24565$$

$$C_{\frac{5}{2}}^{(6)} = \frac{1}{a'} \times 23{,}14387 \qquad C_{\frac{5}{2}}^{(10)} = \frac{1}{a'} \times 10{,}39741 \qquad C_{\frac{5}{2}}^{(14)} = \frac{1}{a'} \times 4{,}12790$$

$$C_{\frac{5}{2}}^{(7)} = \frac{1}{a'} \times 19{,}25046 \qquad C_{\frac{5}{2}}^{(11)} = \frac{1}{a'} \times 8{,}32969 \qquad C_{\frac{5}{2}}^{(15)} = \frac{1}{a'} \times 3{,}23120$$

$$C_{\frac{5}{2}}^{(8)} = \frac{1}{a'} \times 15{,}82608 \qquad C_{\frac{5}{2}}^{(12)} = \frac{1}{a'} \times 6{,}62955 \qquad C_{\frac{5}{2}}^{(16)} = \frac{1}{a'} \times 2{,}51561$$

39. Making $s = \frac{5}{2}$, $C_{\frac{7}{2}}^{(k)} = \frac{\sin^2\beta}{10\cos^2\beta} \cdot \left\{ \frac{(2k+5)(2k+3)}{k} C_{\frac{5}{2}}^{(k-1)} - \frac{(2k-5)(2k-3)}{k} C_{\frac{5}{2}}^{(k+1)} \right\}$. Thus we get

$$\mathrm{C}_{\frac{5}{2}}^{(6)} = \frac{1}{a'} \times 221{,}8780 \qquad \mathrm{C}_{\frac{5}{2}}^{(9)} = \frac{1}{a'} \times 143{,}6296 \qquad \mathrm{C}_{\frac{5}{2}}^{(12)} = \frac{1}{a'} \times 84{,}9489$$

$$\mathrm{C}_{\frac{5}{2}}^{(7)} = \frac{1}{a'} \times 194{,}2735 \qquad \mathrm{C}_{\frac{5}{2}}^{(10)} = \frac{1}{a'} \times 121{,}5988 \qquad \mathrm{C}_{\frac{5}{2}}^{(13)} = \frac{1}{a'} \times 70{,}2184$$

$$\mathrm{C}_{\frac{5}{2}}^{(8)} = \frac{1}{a'} \times 167{,}9770 \qquad \mathrm{C}_{\frac{5}{2}}^{(11)} = \frac{1}{a'} \times 102{,}0404 \qquad \mathrm{C}_{\frac{5}{2}}^{(14)} = \frac{1}{a'} \times 57{,}6762$$

$$\mathrm{C}_{\frac{5}{2}}^{(15)} = \frac{1}{a'} \times 47{,}1003$$

40. Making $s = \frac{7}{2}$, $\mathrm{C}_{\frac{7}{2}}^{(k)} = \frac{\sin^2\beta}{14\cos^2\beta}\left\{\frac{(2k+7)(2k+5)}{k}\,\mathrm{C}_{\frac{5}{2}}^{(k-1)} - \frac{(2k-7)(2k-5)}{k}\,\mathrm{C}_{\frac{5}{2}}^{(k+1)}\right\}$. From this,

$$\mathrm{C}_{\frac{7}{2}}^{(7)} = \frac{1}{a'} \times 1830{,}596 \qquad \mathrm{C}_{\frac{7}{2}}^{(10)} = \frac{1}{a'} \times 1266{,}709 \qquad \mathrm{C}_{\frac{7}{2}}^{(13)} = \frac{1}{a'} \times 807{,}945$$

$$\mathrm{C}_{\frac{7}{2}}^{(8)} = \frac{1}{a'} \times 1636{,}049 \qquad \mathrm{C}_{\frac{7}{2}}^{(11)} = \frac{1}{a'} \times 1099{,}213 \qquad \mathrm{C}_{\frac{7}{2}}^{(14)} = \frac{1}{a'} \times 685{,}214$$

$$\mathrm{C}_{\frac{7}{2}}^{(9)} = \frac{1}{a'} \times 1446{,}655 \qquad \mathrm{C}_{\frac{7}{2}}^{(12)} = \frac{1}{a'} \times 946{,}016$$

41. Making $s = \frac{9}{2}$, $\mathrm{C}_{\frac{9}{2}}^{(k)} = \frac{\sin^2\beta}{18\cos^2\beta}\left\{\frac{(2k+9)(2k+7)}{k}\,\mathrm{C}_{\frac{7}{2}}^{(k-1)} - \frac{(2k-9)(2k-7)}{k}\,\mathrm{C}_{\frac{7}{2}}^{(k+1)}\right\}$. From this,

$$\mathrm{C}_{\frac{9}{2}}^{(8)} = \frac{1}{a'} \times 15366{,}90 \qquad \mathrm{C}_{\frac{9}{2}}^{(10)} = \frac{1}{a'} \times 12473{,}68 \qquad \mathrm{C}_{\frac{9}{2}}^{(12)} = \frac{1}{a'} \times 9786{,}59$$

$$\mathrm{C}_{\frac{9}{2}}^{(9)} = \frac{1}{a'} \times 13907{,}74 \qquad \mathrm{C}_{\frac{9}{2}}^{(11)} = \frac{1}{a'} \times 11092{,}76 \qquad \mathrm{C}_{\frac{9}{2}}^{(13)} = \frac{1}{a'} \times 8570{,}07$$

SECTION 13.

Numerical calculation of (0,1) $\mathrm{C}_s^{(k)}$, (1,0) $\mathrm{C}_s^{(k)}$, *&c.*

42. It will be sufficient to form, by differentiation, the expression for one of the differential coefficients of each order, as the others can then be derived by simple addition. For $\mathrm{C}_s^{(k)}$ is a function of a' and a of -1 dimension: hence

$a' \frac{d}{d a'} \mathrm{C}_s^{(k)} + a \frac{d}{d a} \mathrm{C}_s^{(k)} = - \mathrm{C}_s^{(k)}$, or $(1,0)\, \mathrm{C}_s^{(k)} + (0,1)\, \mathrm{C}_s^{(k)} = - \mathrm{C}_s^{(k)}$. Again (as another instance) $\frac{d^4}{d a'^3 . d a} \mathrm{C}_s^{(k)}$ is a function of a' and a of -5 dimensions; consequently $a' \frac{d^5}{d a'^4 . d a} \mathrm{C}_s^{(k)} + a \frac{d^5}{d a'^3 . d a^2} \mathrm{C}_s^{(k)} = - 5 \frac{d^4}{d a'^3 . d a} \mathrm{C}_s^{(k)}$: or, multiplying both sides by $a'^3 a$, $(4,1)\, \mathrm{C}_s^{(k)} + (3,2)\, \mathrm{C}_s^{(k)} = - 5\, (3,1)\, \mathrm{C}_s^{(k)}$. It is indifferent which coefficient of each order we calculate first; and for the algebraical process it is rather most convenient to differentiate successively with regard to the same quantity (as a').

43. Now $\frac{d}{d a'} \left\{ \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^s} \right\} =$

$$- \frac{1}{2} \cdot \frac{1}{a'} \cdot \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^s}$$

$$+ s \left(- \frac{1}{a} + \frac{a}{a'^2} \right) \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^{s+1}}$$

or, taking the coefficient of $\cos k \chi$ in the expansion on both sides,

$$\frac{d}{d a'} \mathrm{C}_s^{(k)} = - \frac{1}{2} \cdot \frac{1}{a'} \mathrm{C}_s^{(k)} + \left(- \frac{1}{a} + \frac{a}{a'^2} \right) s . \mathrm{C}_{s+1}^{(k)}$$

Differentiating this formula with respect to a', and using the same formula to simplify the differential coefficient, we get $\frac{d^2}{d a'^2} \mathrm{C}_s^{(k)}$. In the same manner $\frac{d^3}{d a'^3} \mathrm{C}_s^{(k)}$, &c. are found; multiplying them $\left(\text{beginning with } \frac{d}{d a'} \mathrm{C}_s^{(k)}\right)$ by a', a'^2, a'^3, &c., we obtain the following expressions:

$$(1,0)\, \mathrm{C}_s^{(k)} = - \frac{1}{2} \mathrm{C}_s^{(k)} + \left(- \frac{1}{\alpha} + \alpha \right) s . \mathrm{C}_{s+1}^{(k)}$$

$$(2,0)\, \mathrm{C}_s^{(k)} = + \frac{3}{4} \mathrm{C}_s^{(k)} + \left(\frac{1}{\alpha} - 3 \alpha \right) s . \mathrm{C}_{s+1}^{(k)} + \left(- \frac{1}{\alpha} + \alpha \right)^2 . s . \overline{s+1} . \mathrm{C}_{s+2}^{(k)}$$

$$(3,0)\, \mathrm{C}_s^{(k)} = - \frac{15}{8} \mathrm{C}_s^{(k)} + \frac{9}{4} \left(- \frac{1}{\alpha} + 5 \alpha \right) s . \mathrm{C}_{s+1}^{(k)}$$

$$+\frac{3}{2}\left(\frac{1}{\alpha}-\alpha\right)\left(-\frac{1}{\alpha}+5\alpha\right).s.\overline{s+1}.\mathrm{C}_{s+2}^{(k)}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^{3}.s.\overline{s+1}.\overline{s+2}.\mathrm{C}_{s+3}^{(k)}$$

$$(4,0)\ \mathrm{C}_{s}^{(k)}=+\frac{105}{16}\,\mathrm{C}_{s}^{(k)}+\frac{15}{2}\left(\frac{1}{\alpha}-7\alpha\right)s.\mathrm{C}_{s+1}^{(k)}$$

$$+\left\{\left(\frac{9}{2}.\frac{1}{\alpha}-\frac{15}{2}\alpha\right).\left(\frac{1}{\alpha}-7\alpha\right)-6\right\}.s.\overline{s+1}.\mathrm{C}_{s+2}^{(k)}$$

$$+2\left(-\frac{1}{\alpha}+\alpha\right)^{2}.\left(\frac{1}{\alpha}-7\alpha\right).s.\overline{s+1}.\overline{s+2}.\mathrm{C}_{s+3}^{(k)}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^{4}.s.\overline{s+1}.\overline{s+2}.\overline{s+3}.\mathrm{C}_{s+4}^{(k)}$$

$$(5,0)\ \mathrm{C}_{s}^{(k)}=-\frac{945}{32}\,\mathrm{C}_{s}^{(k)}+\frac{525}{16}\left(-\frac{1}{\alpha}+9\alpha\right)s.\mathrm{C}_{s+1}^{(k)}$$

$$+\frac{75}{4}\left\{-\left(\frac{1}{\alpha}-3\alpha\right)\left(\frac{1}{\alpha}-7\alpha\right)+4\right\}s.\overline{s+1}.\mathrm{C}_{s+2}^{(k)}$$

$$+\frac{15}{2}\left(\frac{1}{\alpha}-\alpha\right)\left\{-\left(\frac{1}{\alpha}-3\alpha\right)\left(\frac{1}{\alpha}-7\alpha\right)+4\right\}s.\overline{s+1}.\overline{s+2}.\mathrm{C}_{s+3}^{(k)}$$

$$+\frac{5}{2}\left(\frac{1}{\alpha}-\alpha\right)^{3}.\left(-\frac{1}{\alpha}+9\alpha\right)s.\overline{s+1}.\overline{s+2}.\overline{s+3}.\mathrm{C}_{s+4}^{(k)}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^{5}.s.\overline{s+1}.\overline{s+2}.\overline{s+3}.\overline{s+4}.\mathrm{C}_{s+5}^{(k)}$$

44. Using the same value of α as before, and making $s=\frac{1}{2}$, these expressions become,

$$(1,0)\ \mathrm{C}_{\frac{1}{2}}^{(k)}=-\frac{1}{2}\,\mathrm{C}_{\frac{1}{2}}^{(k)}-0{,}3295790\,.\,\mathrm{C}_{\frac{3}{2}}^{(k)}$$

$$(2,0)\ \mathrm{C}_{\frac{1}{2}}^{(k)}=+\frac{3}{4}\,\mathrm{C}_{\frac{1}{2}}^{(k)}-0{,}3937533\,.\,\mathrm{C}_{\frac{3}{2}}^{(k)}+0{,}3258670\,.\,\mathrm{C}_{\frac{5}{2}}^{(k)}$$

$$(3,0)\ \mathrm{C}_{\frac{1}{2}}^{(k)}=-\frac{15}{8}\,\mathrm{C}_{\frac{1}{2}}^{(k)}+2{,}5134426\,.\,\mathrm{C}_{\frac{3}{2}}^{(k)}+1{,}6567557\,.\,\mathrm{C}_{\frac{5}{2}}^{(k)}-0{,}5369945\,.\,\mathrm{C}_{\frac{7}{2}}^{(k)}$$

$$(4,0)\ C_{\frac{1}{2}}^{(k)} = +\frac{105}{16}\,C_{\frac{1}{2}}^{(k)} - 13{,}8031342\,.\,C_{\frac{3}{2}}^{(k)} - 6{,}6980500\,.\,C_{\frac{5}{2}}^{(k)}$$

$$- 5{,}9973139\,.\,C_{\frac{7}{2}}^{(k)} + 1{,}2388750\,.\,C_{\frac{9}{2}}^{(k)}$$

$$(5,0)\ C_{\frac{1}{2}}^{(k)} = -\frac{945}{32}\,C_{\frac{1}{2}}^{(k)} + 84{,}1230534\,.\,C_{\frac{3}{2}}^{(k)} + 15{,}4872787\,.\,C_{\frac{5}{2}}^{(k)}$$

$$+ 10{,}2085636\,.\,C_{\frac{7}{2}}^{(k)} + 24{,}0925995\,.\,C_{\frac{9}{2}}^{(k)} - 3{,}6747654\,.\,C_{\frac{11}{2}}^{(k)}$$

45. Making $s = \frac{3}{2}$, the formulæ give

$$(1,0)\ C_{\frac{3}{2}}^{(k)} = -\frac{1}{2}\,C_{\frac{3}{2}}^{(k)} - 0{,}9887370\,.\,C_{\frac{5}{2}}^{(k)}$$

$$(2,0)\ C_{\frac{3}{2}}^{(k)} = +\frac{3}{4}\,C_{\frac{3}{2}}^{(k)} - 1{,}1812599\,.\,C_{\frac{5}{2}}^{(k)} + 1{,}6293350\,.\,C_{\frac{7}{2}}^{(k)}$$

$$(3,0)\ C_{\frac{3}{2}}^{(k)} = -\frac{15}{8}\,C_{\frac{3}{2}}^{(k)} + 7{,}5403278\,.\,C_{\frac{5}{2}}^{(k)} + 8{,}2837785\,.\,C_{\frac{7}{2}}^{(k)} - 3{,}7589615\,.\,C_{\frac{9}{2}}^{(k)}$$

46. Making $s = \frac{5}{2}$, the first formula gives

$$(1,0)\ C_{\frac{5}{2}}^{(k)} = -\frac{1}{2}\,C_{\frac{5}{2}}^{(k)} - 1{,}6478950\,.\,C_{\frac{7}{2}}^{(k)}.$$

47. Substituting in these the values of $C_{\frac{1}{2}}^{(k)}$, $C_{\frac{3}{2}}^{(k)}$, &c. found in the last section for different values of k, we form the following tables:

For the development of the first term.

$k = 8$

$$(0,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 0{,}0414571$$

$$(1,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -0{,}414243$$

$$(2,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 4{,}71815$$

$$(3,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -61{,}0595$$

$(4,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 897,236$

$(5,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -14993,97$

$k = 9$ $(0,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0,0283925$

$(1,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -0,312594$ $(0,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0,284001$

$(2,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 3,86300$ $(1,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -3,23821$

$(3,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -53,5643$ $(2,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 41,9753$

$(4,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 832,244$ $(3,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -617,987$

$(5,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -14512,93$ $(4,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 10351,71$

$k = 10$ $(0,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0,0195495$

$(1,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -0,234856$ $(0,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0,215306$

$(2,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 3,13393$ $(1,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -2,66422$ $(0,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 2,23361$

$(3,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -46,3921$ $(2,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 36,9903$ $(1,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -28,9976$

$(4,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 761,088$ $(3,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -575,520$ $(2,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 427,559$

$(5,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -13860,27$ $(4,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 10054,83$ $(3,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -7177,23$

$k = 11$ $(0,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0,0135189$

$(1,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -0,176097$ $(0,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0,162578$

$(2,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 2,52220$ $(1,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -2,17001$ $(0,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 1,84485$

$(3,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -39,7288$ $(2,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 32,1622$ $(1,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -25,6522$

$(0,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 20,1176$ $(4,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 687,024$ $(3,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -528,109$

$(2,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 399,460$ $(1,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -296,851$ $(5,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -13066,87$

$(4,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 9631{,}75$ $(3,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times - 6991{,}20$ $(2,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 4993{,}90$

$k = 12$ $(0,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0{,}0093812$

$(1,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 0{,}131750$ $(0,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0{,}122369$

$(2,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 2{,}01559$ $(1,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 1{,}75209$ $(0,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 1{,}50735$

$(3,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 33{,}6822$ $(2,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 27{,}6354$ $(1,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 22{,}3791$

$(0,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 17{,}8570$ $(4,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 612{,}866$ $(3,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 478{,}137$

$(2,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 367{,}595$ $(1,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 278{,}079$ $(0,4)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 206{,}651$

$(5,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 12169{,}39$ $(4,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 9105{,}06$ $(3,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 6714{,}37$

$(2,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 4876{,}40$ $(1,4)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times - 3486{,}00$

$k = 13$ $(0,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0{,}0065274$

$(1,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 0{,}098398$ $(0,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0{,}091871$

$(2,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 1{,}60062$ $(1,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 1{,}40382$ $(0,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 1{,}22008$

$(3,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 28{,}3028$ $(2,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 23{,}5009$ $(1,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 19{,}2894$

$(0,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 15{,}6292$ $(4,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 540{,}744$ $(3,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 427{,}533$

$(2,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 333{,}529$ $(1,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 256{,}371$ $(0,4)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 193{,}854$

$(5,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 11205{,}35$ $(4,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 8501{,}63$ $(3,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 6363{,}96$

$(2,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 4696{,}32$ $(1,4)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times - 3414{,}46$ $(0,5)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 2445{,}19$

For the development of the second term,

$k = 9$ $(0,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0{,}904785$

$(1,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times - 13{,}18976$

$(2,0)\ C^{(9)}_{\frac{3}{2}} = \frac{1}{a'} \times 219,4819$

$(3,0)\ C^{(9)}_{\frac{3}{2}} = \frac{1}{a'} \times -4152,686$

$k = 10$ $(0,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,682935$

$(1,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -10,62177$ $(0,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 9,93883$

$(2,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 186,3554$ $(1,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -165,1119$

$(3,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -3677,095$ $(2,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 3118,029$

$k = 11$ $(0,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,513799$

$(1,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -8,49277$ $(0,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 7,97897$

$(2,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 156,8038$ $(1,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -139,8183$ $(0,2)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 123,8604$

$(3,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -3224,776$ $(2,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 2754,365$ $(1,2)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -2334,910$

$k = 12$ $(0,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,385521$

$(1,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -6,74764$ $(0,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 6,36212$

$(2,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 130,8682$ $(1,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -117,3729$ $(0,2)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 104,6487$

$(3,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -2803,076$ $(2,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 2410,471$ $(1,2)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -2058,352$

$(0,3)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 1744,406$

For the development of the third term,

$k = 10$ $(0,0)\ C^{(10)}_{\frac{5}{2}} = \frac{1}{a'} \times 10,39741$

$(1,0)\ C^{(10)}_{\frac{5}{2}} = \frac{1}{a'} \times -205,5808$

$k = 11$ $(0,0)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times 8,32969$

$(1,0)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times -172,3167$ $(0,1)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times 163,9870$

48. Employing these numbers in the calculation of $L^{(8)}$, $L^{(9)}$, &c. $M^{(9)}$, $M^{(10)}$, &c. $N^{(10)}$ and $N^{(11)}$, from the expressions in (20), (24), and (28), we obtain the following numerical values:

$$L^{(8)} = \frac{m}{a'} \times -333{,}0969$$

$$L^{(9)} = \frac{m}{a'} \times 1273{,}4929$$

$$L^{(10)} = \frac{m}{a'} \times -1945{,}7913$$

$$L^{(11)} = \frac{m}{a'} \times 1485{,}3152$$

$$L^{(12)} = \frac{m}{a'} \times -566{,}5632$$

$$L^{(13)} = \frac{m}{a'} \times 86{,}3635$$

$$M^{(9)} = \frac{m}{a'} \times -503{,}4795$$

$$M^{(10)} = \frac{m}{a'} \times 1088{,}9148$$

$$M^{(11)} = \frac{m}{a'} \times -787{,}0581$$

$$M^{(12)} = \frac{m}{a'} \times 190{,}0487$$

$$N^{(10)} = \frac{m}{a'} \times -85{,}3347$$

$$N^{(11)} = \frac{m}{a'} \times 58{,}8603$$

49. The computation of these quantities has been effected by means of algebraical operations of great complexity, and numerical calculations of no inconsiderable length; and it is not easy to find in the operations themselves any verification of their accuracy. This has imposed on me the necessity of examining closely every line of figures before I proceeded to another. I have

had the advantage however of comparing the calculated values several times with the values which I calculated nearly four years ago. At that time I developed the principal fraction in a different manner, and I expressed the quantities $C^{(k)}_{\frac{1}{2}}$ &c. by different formulæ; and the fundamental number differed by a few units in the last place of decimals. The numbers admitted of comparison at several intermediate points before arriving at the final results; and one small error was discovered in the old calculations, and one in the new ones. Upon the whole, I am certain that there is no error of importance in these numbers; and I think it highly probable that there is no error, except such as inevitably arise from the rejection of figures beyond a certain place of decimals. It is impossible to assert that the last figure preserved is correct, or even the last but one; but I do not think that the last but two is wrong.

SECTION 14.

Numerical calculation of the long inequality in the epoch, depending on (13 × mean long. Earth − 8 × mean long. Venus).

50. The most convenient form in which the expression of (29) can be put is the following.

$$\Big\{ L^{(8)} . e'^5 . \cos(5\varpi') + L^{(9)} . e'^4 e . \cos(4\varpi' + \varpi) + L^{(10)} . e'^3 e^2 . \cos(3\varpi' + 2\varpi)$$

$$+ L^{(11)} . e'^2 e^3 . \cos(2\varpi' + 3\varpi) + L^{(12)} . e' e^4 . \cos(\varpi' + 4\varpi)$$

$$+ L^{(13)} . e^5 . \cos(5\varpi) + M^{(9)} . e'^3 f^2 . \cos(3\varpi' + 2\theta)$$

$$+ M^{(10)} e'^2 e f^2 . \cos(2\varpi' + \varpi + 2\theta) + M^{(11)} . e' e^2 f^2 . \cos(\varpi' + 2\varpi + 2\theta)$$

$$+ M^{(12)} . e^3 f^2 . \cos(3\varpi + 2\theta) + N^{(10)} . e' f^4 . \cos(\varpi' + 4\theta)$$

$$+ N^{(11)} . e f^4 . \cos(\varpi + 4\theta) \Big\} \cos\{13(n't + \epsilon') - 8(nt + \epsilon)\}$$

$$+ \Big\{ L^{(8)} . e'^5 . \sin(5\varpi') + L^{(9)} . e'^4 e . \sin(4\varpi' + \varpi) + L^{(10)} . e'^3 e^2 . \sin(3\varpi' + 2\varpi)$$

$$+ L^{(11)} . e'^2 e^3 . \sin(2\varpi' + 3\varpi) + L^{(12)} . e' e^4 . \sin(\varpi' + 4\varpi)$$

$$+ \overset{(13)}{\mathrm{L}} . e^5 . \sin (5 \varpi) + \overset{(9)}{\mathrm{M}} . e'^3 f^2 . \sin (3 \varpi' + 2 \theta)$$

$$+ \overset{(10)}{\mathrm{M}} . e'^2 e f^2 . \sin (2 \varpi' + \varpi + 2 \theta) + \overset{(11)}{\mathrm{M}} . e' e^2 f^2 . \sin (\varpi' + 2 \varpi + 2 \theta)$$

$$+ \overset{(12)}{\mathrm{M}} . e^3 f^2 . \sin (3 \varpi + 2 \theta) + \overset{(10)}{\mathrm{N}} . e' f^4 . \sin (\varpi' + 4 \theta)$$

$$+ \overset{(11)}{\mathrm{N}} . e f^4 . \sin (\varpi + 4 \theta) \Big\} . \sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

The elements e', e, &c. are all subject to small permanent variation; and (considering the great length of period of the inequality which we are calculating,) those variations may have a sensible influence upon it. It is prudent therefore, as well as interesting, to take into account these variations.

51. Let P and Q be the values of the coefficients of $\cos \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$ and $\sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$ in the expression above, giving to the elements the values which they had in 1750. Then, as all the permanent variations are small, the powers of t above the first may be rejected, and the coefficients at the time t after 1750 may be represented by $\mathrm{P} + pt$ and $\mathrm{Q} + qt$. Thus the term of R becomes

$$(\mathrm{P} + pt) \cos \{13 (n't + \epsilon') - 8 (nt + \epsilon)\} + (\mathrm{Q} + qt) \sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\};$$

and by (2), omitting the terms depending on $\frac{d\,\mathrm{R}}{d\,a'}$ and $\frac{d\,\mathrm{R}}{d\,e'}$ for the reasons in (31),

$$\frac{d\,n'}{d\,t} = - \frac{39\, n'^2 a'}{\mu'} (\mathrm{P} + pt) \sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

$$+ \frac{39\, n'^2 a'}{\mu'} (\mathrm{Q} + qt) \cos \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

$$\frac{d\,\epsilon'}{d\,t} = + \frac{39\, n'^2 a'}{\mu'} (\mathrm{P}t + pt^2) \sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

$$- \frac{39\, n'^2 a'}{\mu'} (\mathrm{Q}t + qt^2) \cos \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

Integrating these, (considering n', ϵ', n, and ϵ, on the right-hand side, as constants,) and substituting in the expression $n't + \epsilon'$, it becomes

$$\mathrm{N}'t + \mathrm{E}'$$

$$+ \frac{39\, n'^2 a'}{\mu'} \left\{ \frac{\mathrm{P} + pt}{(13 n' - 8 n)^2} + \frac{2q}{(13 n' - 8 n)^3} \right\} \sin \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$$

$$+\frac{39\,n'^2 a'}{\mu'}\left\{\frac{-Q-q\,t}{(13\,n'-8\,n)^2}+\frac{2\,p}{(13\,n'-8\,n)^3}\right\}\cos\{13\,(n'\,t+\epsilon')-8\,(n\,t+\epsilon)\}$$

The terms added to $N'\,t + E'$ constitute the inequality in the epoch.

52. The values of the elements for 1750 and their annual variations are given by **Laplace** in the Mécanique Céleste, 2^{me} Partie, Livre 6, N^{os} 22 and 26. To give them the form necessary for our purpose, we must from the variation in a Julian year deduce the variation for a unit of time. Now a Julian year is (nearly) the time in which the angle $n'\,t$ increases by $2\,\pi$; its expression is therefore $\frac{2\,\pi}{n'}$. Consequently if we multiply the annual variations by $\frac{n'}{2\,\pi}$, we shall have the variations in a unit of time: and if we multiply them by $\frac{n'\,t}{2\,\pi}$, we shall have the variations in the time t. With regard to the quantities μ', &c. introduced by **Laplace** for the purpose of altering his assumed masses if necessary, it may be observed that the only planet which materially affects the changes of the elements, and whose mass is known with certainty to require a change, is Venus herself. The investigations of **Burckhardt** and **Bessel** lead to the same conclusion as my own (Phil. Trans. 1828), namely, that the mass of Venus is $\frac{8}{9}$ × the mass assumed by **Delambre**, or $\frac{1}{401211}$ of the sun's mass. **Laplace** supposed it $\frac{1+\mu'}{383137}$ of the sun's mass: the comparison of these gives **Laplace**'s $\mu' = -\,,045$. In using **Laplace**'s expressions, therefore, I shall suppose $\mu' = -\,,045$, and $\mu, \mu'', \mu''', \&c. = 0$. For convenience, the centesimal * division will be retained.

53. Thus we have

$$n = \frac{650198000}{399993009} \times n'$$
$$e' = 0{,}01681395 - 0{,}0000000729 \times n'\,t$$
$$e = 0{,}00688405 - 0{,}0000001005\,\dagger \times n'\,t$$
$$f = 0{,}02960597 + 0{,}0000000172 \times n'\,t$$
$$\varpi' = 109^{g}{,}5790 + 0{,}0000091017 \times n'\,t$$

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* **Borda**'s tables, published by **Delambre**, have been used in these computations.

† The variations of the elements of Venus do not agree with those of **Lindenau**'s tables.

$$\varpi = 142^{g},1241 - 0,0000018080 \times n' t$$
$$\theta = 82^{g},7093 - 0,0000139997 \times n' t$$

The node and inclination are those on the earth's *true* orbit. All the coefficients of $n' t$ are in decimal parts of the radius 1, and not in parts of a degree.

54. From these we deduce the following values, the figures within the brackets being the logarithms of the numbers.

$$e'^5 = + (91,1283485) - (86,46438) \,.\, n' t$$
$$e'^4 e = + (90,7405229) - (86,24488) \,.\, n' t$$
$$e'^3 e^2 = + (90,3526973) - (85,97806) \,.\, n' t$$
$$e'^2 e^3 = + (89,9648717) - (85,68477) \,.\, n' t$$
$$e' e^4 = + (89,5770461) - (85,37453) \,.\, n' t$$
$$e^5 = + (89,1892205) - (85,05252) \,.\, n' t$$
$$e'^3 f^2 = + (91,6197677) - (86,69331) \,.\, n' t$$
$$e'^2 e f^2 = + (91,2319421) - (86,57650) \,.\, n' t$$
$$e' e^2 f^2 = + (90,8441165) - (86,35426) \,.\, n' t$$
$$e^3 f^2 = + (90,4562909) - (86,08606) \,.\, n' t$$
$$e' f^4 = + (92,1111869) - (86,41479) \,.\, n' t$$
$$e f^4 = + (91,7233613) - (86,81239) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(8)}}{m} \,.\, \cos (5 \varpi') = + (2,3572098) + (98,04404) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(8)}}{m} \,.\, \sin (5 \varpi') = - (2,3859510) + (98,01530) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(9)}}{m} \,.\, \cos (4 \varpi' + \varpi) = - (3,0841670) - (98,12469) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(9)}}{m} \,.\, \sin (4 \varpi' + \varpi) = + (2,5856285) - (98,62323) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(10)}}{m} \,.\, \cos (3 \varpi' + 2 \varpi) = + (3,2799989) - (97,97020) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(10)}}{m} \,.\, \sin (3 \varpi' + 2 \varpi) = + (2,5956493) + (98,65455) \,.\, n' t$$

$$\frac{a' \, \mathrm{L}^{(11)}}{m} \,.\, \cos (2 \varpi' + 3 \varpi) = - (3,0497482) + (98,09507) \,.\, n' t$$

$$\frac{a' \mathrm{L}^{(11)}}{m} . \sin (2 \varpi' + 3 \varpi) = - (2{,}9885633) - (98{,}15626) . n' t$$

$$\frac{a' \mathrm{L}^{(12)}}{m} . \cos (\varpi' + 4 \varpi) = + (2{,}2816808) - (96{,}99874) . n' t$$

$$\frac{a' \mathrm{L}^{(12)}}{m} . \sin (\varpi' + 4 \varpi) = + (2{,}7269678) + (96{,}55345) . n' t$$

$$\frac{a' \mathrm{L}^{(13)}}{m} . \cos (5 \varpi) = + (1{,}1565787) - (96{,}88643) . n' t$$

$$\frac{a' \mathrm{L}^{(13)}}{m} . \sin (5 \varpi) = - (1{,}9302586) - (96{,}11275) . n' t$$

$$\frac{a' \mathrm{M}^{(9)}}{m} . \cos (3 \varpi' + 2 \theta) = - (1{,}6642318) - (96{,}54170) . n' t$$

$$\frac{a' \mathrm{M}^{(9)}}{m} . \sin (3 \varpi' + 2 \theta) = - (2{,}7001492) + (95{,}50578) . n' t$$

$$\frac{a' \mathrm{M}^{(10)}}{m} . \cos (2 \varpi' + \varpi + 2 \theta) = - (2{,}6468283) + (98{,}06223) . n' t$$

$$\frac{a' \mathrm{M}^{(10)}}{m} . \sin (2 \varpi' + \varpi + 2 \theta) = + (2{,}9976206) + (97{,}71143) . n' t$$

$$\frac{a' \mathrm{M}^{(11)}}{m} . \cos (\varpi' + 2 \varpi + 2 \theta) = + (2{,}8001796) - (98{,}02467) . n' t$$

$$\frac{a' \mathrm{M}^{(11)}}{m} . \sin (\varpi' + 2 \varpi + 2 \theta) = - (2{,}6722198) - (98{,}15263) . n' t$$

$$\frac{a' \mathrm{M}^{(12)}}{m} . \cos (3 \varpi + 2 \theta) = - (2{,}2752440) + (96{,}91212) . n' t$$

$$\frac{a' \mathrm{M}^{(12)}}{m} . \sin (3 \varpi + 2 \theta) = + (1{,}3880761) + (97{,}79929) . n' t$$

$$\frac{a' \mathrm{N}^{(10)}}{m} . \cos (\varpi' + 4 \theta) = - (1{,}8370062) - (97{,}37538) . n' t$$

$$\frac{a' \mathrm{N}^{(10)}}{m} . \sin (\varpi' + 4 \theta) = - (1{,}7042256) + (97{,}50816) . n' t$$

$$\frac{a' \mathrm{N}^{(11)}}{m} . \cos (\varpi + 4\,\theta) = + (1{,}3847917) + (97{,}49139) . n' t$$

$$\frac{a' \mathrm{N}^{(11)}}{m} . \sin (\varpi + 4\,\theta) = + (1{,}7294138) - (97{,}14677) . n' t$$

55. Substituting these in the expressions of (50), we find

$$\mathrm{P} = - \frac{m}{a'} \times (94{,}1302623) \qquad p = + \frac{m}{a'} \times (89{,}08397) . n'$$

$$\mathrm{Q} = - \frac{m}{a'} \times (94{,}0722348) \qquad q = + \frac{m}{a'} \times (89{,}47976) . n'$$

and making $\frac{m}{\mu'} = \frac{1}{401211}$, and $13\, n' - 8\, n = - \frac{1674883}{399993090} \times n'$, in the expression of (51), we find for the long inequality

$$\begin{aligned} &\{- (94{,}8787039) + n' t \times (89{,}82780)\} . \sin \{13\, (n' t + \epsilon') - 8\, (n t + \epsilon)\} \\ &+ \{+ (94{,}8139258) - n' t \times (90{,}22359)\} . \cos \{13\, (n' t + \epsilon') - 8\, (n t + \epsilon)\} \end{aligned}$$

which may be put in the form

$$\begin{aligned} &\{+ (94{,}9992364) - n' t \times (90{,}20461)\} . \sin \{8\, (n t + \epsilon) - 13\, (n' t + \epsilon') \\ &\qquad + 40^\circ\, 44'\, 34'' - n' t \times (94{,}91918)\} \end{aligned}$$

where the degrees, &c. in the argument are sexagesimal. The coefficient is expressed by a multiple of the radius: to express the principal term in sexagesimal seconds, it must be divided by $\sin 1''$. And if Y be the number of years after 1750, since $n' t =$ mean motion of the earth in Y years $= 2\,\pi . \mathrm{Y} = 6 . 60^3 . \mathrm{Y}$ in seconds, the coefficients of $n' t$ must be multiplied by $6 . 60^3 . \mathrm{Y}$, and their values will then be exhibited in sexagesimal seconds. Thus we find at length for the inequality

$$\begin{aligned} &\{2'',059 - \mathrm{Y} \times 0'',0002076\} \times \sin \{8\, (n t + \epsilon) - 13\, (n' t + \epsilon') \\ &\qquad + 40^\circ\, 44'\, 34'' - \mathrm{Y} \times 10'',76\}. \end{aligned}$$

56. The mean longitudes $n t + \epsilon$, $n' t + \epsilon'$, are measured from the equinox of 1750. But if l, l', are the mean longitudes of Venus and the Earth measured from the place of the equinox Y years after 1750, then (in consequence of precession)

$$n t + \epsilon = l - \mathrm{Y} \times 50'',1$$
$$n' t + \epsilon' = l' - \mathrm{Y} \times 50'',1$$

Consequently $8\,(n t + \epsilon) - 13\,(n' t + \epsilon') = 8\,l - 13\,l' + \mathrm{Y} \times 250'',5$.

Substituting this, the expression for the inequality is

$$\{2'',059 - \mathrm{Y} \times 0'',0002076\} \times \sin\{8\,l - 13\,l' + 40^\circ\,44'\,34'' + \mathrm{Y} \times 239'',7\}$$

57. I have compared the calculations of the principal part of this inequality with the calculations made in 1827. Two errors were discovered in the former calculations, one of which was important. I am quite confident that there is no sensible error in the results now presented. The terms depending on Y were not calculated on the former occasion: but the calculations now made have been carefully revised.

SECTION 15.

Numerical calculation of the long inequality in the length of the axis major.

58. This being very small, we shall omit the variable terms. Thus we have

$$\frac{d\,a'}{d\,t} = + \frac{26\,n'\,a'^2}{\mu'}\,\mathrm{P} \,.\, \sin\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$
$$- \frac{26\,n'\,a'^2}{\mu}\,\mathrm{Q} \,.\, \cos\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$

whence

$$a' = \mathrm{A}' - \frac{26\,n'\,a'}{13\,n' - 8\,n} \,.\, \frac{\mathrm{P}\,a'}{\mu} \,.\, \cos\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$
$$- \frac{26\,n'\,a'}{13\,n' - 8\,n} \,.\, \frac{\mathrm{Q}\,a'}{\mu'} \,.\, \sin\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$
$$= \mathrm{A}' - a' \,.\, (92,31993) \,.\, \cos\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$
$$- a' \,.\, (92,26190)\, \sin\{13\,(n' t + \epsilon') - 8\,(n t + \epsilon)\}$$
$$= \mathrm{A}' - a' \times 0,000000027756 \times \cos\{8\,(n t + \epsilon) - 13\,(n' t + \epsilon') + 41^\circ\,11'\}$$

The magnitude of the coefficient is barely $\frac{1}{40}$th of LAPLACE's minimum, and this inequality may therefore be neglected.

SECTION 16.

Numerical calculation of the long inequality in the longitude of perihelion.

59. The expression for $\frac{d\varpi'}{dt}$ being $-\frac{n'a'}{\mu' e'}\cdot\frac{dR}{de'}$, the part which we have to consider may be put under the form

$$\begin{aligned}\frac{d\varpi'}{dt} = -\frac{n'a'}{\mu' e'^2}\cdot\Big\{ & 5\,\mathrm{L}^{(8)}.e'^5.\cos(5\varpi') + 4\,\mathrm{L}^{(9)}.e'^4 e.\cos(4\varpi'+\varpi) \\ & + 3\,\mathrm{L}^{(10)}.e'^3 e^2.\cos(3\varpi'+2\varpi) + 2\,\mathrm{L}^{(11)}.e'^2 e^3.\cos(2\varpi'+3\varpi) \\ & + \mathrm{L}^{(12)}.e' e^4.\cos(\varpi'+4\varpi) + 3\,\mathrm{M}^{(9)}.e'^3 f^2.\cos(3\varpi'+2\theta) \\ & + 2\,\mathrm{M}^{(10)}.e'^2 e f^2.\cos(2\varpi'+\varpi+2\theta) + \mathrm{M}^{(11)}.e' e^2 f^2.\cos(\varpi'+2\varpi+2\theta) \\ & + \mathrm{N}^{(10)}.e' f^4.\cos(\varpi'+4\theta)\Big\}\cos\Big\{13(n't+\epsilon')-8(nt+\epsilon)\Big\} \\ -\frac{n'a'}{\mu' e'^2}\Big\{ & 5\,\mathrm{L}^{(8)}.e'^5.\sin(5\varpi') + 4\,\mathrm{L}^{(9)}.e'^4 e.\sin(4\varpi'+\varpi) \\ & + 3\,\mathrm{L}^{(10)}.e'^3 e^2.\sin(3\varpi'+2\varpi) + 2\,\mathrm{L}^{(11)}.e'^2 e^3.\sin(2\varpi'+3\varpi) \\ & + \mathrm{L}^{(12)}.e' e^4.\sin(\varpi'+4\varpi) + 3\,\mathrm{M}^{(9)}.e'^3 f^2 \sin(3\varpi'+2\theta) \\ & + 2\,\mathrm{M}^{(10)}.e'^2 e f^2.\sin(2\varpi'+\varpi+2\theta) + \mathrm{M}^{(11)}.e' e^2 f^2.\sin(\varpi'+2\varpi+2\theta) \\ & + \mathrm{N}^{(10)}.e' f^4.\sin(\varpi'+4\theta)\Big\}\sin\Big\{13(n't+\epsilon')-8(nt+\epsilon)\Big\}\end{aligned}$$

which (neglecting the variable terms) is found to equal

$$\begin{aligned} & n'\times(92{,}35866).\cos\{13(n't+\epsilon')-8(nt+\epsilon)\} \\ & \qquad + n'\times(92{,}60190).\sin\{13(n't+\epsilon')-8(nt+\epsilon)\}\end{aligned}$$

Integrating,

$$\begin{aligned}\varpi' = \Pi' & - (94{,}73673).\sin\{13(n't+\epsilon')-8(nt+\epsilon)\} \\ & + (94{,}97997).\cos\{13(n't+\epsilon')-8(nt+\epsilon)\}\end{aligned}$$

or

$$\varpi' = \Pi' + 1'',1250.\sin\{8(nt+\epsilon)-13(n't+\epsilon')\}$$

$$+ 1'',9697 \,.\cos \{8(nt+\varepsilon) - 13(n't+\varepsilon')\}$$

$$= \Pi' + 2'',2683 \,.\sin \{8(nt+\varepsilon) - 13(n't+\varepsilon') + 60^\circ\, 16'\}$$

SECTION 17.

Numerical calculation of the long inequality in the excentricity.

60. On forming the expression for $\frac{de'}{dt}$, or $+ \frac{n'a'}{\mu'e'} \cdot \frac{d\mathrm{R}}{d\varpi'}$, it is immediately seen that the coefficients of $\cos\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$ and $\sin\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$ are related to those above, and that

$$\frac{de'}{dt} = + e'n' \times (92,60190) \,.\cos\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$$
$$- e'n' \times (92,35866) \,.\sin\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$$

Integrating,

$$e' = \mathrm{E}' - e' \times (94,97997) \,.\sin\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$$
$$- e' \times (94,73673) \,.\cos\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$$
$$= \mathrm{E}' - (92,96240) \cos\{8(nt+\varepsilon) - 13(n't+\varepsilon')\}$$
$$+ (93,20564) \,.\sin\{8(nt+\varepsilon) - 13(n't+\varepsilon')\}$$
$$= \mathrm{E}' - 0,0000001849 \,.\cos\{8(nt+\varepsilon) - 13(n't+\varepsilon') + 60^\circ\, 16'\}$$

The principal inequality in the radius vector is that produced by the last term: it is however too small to be sensible.

PART II.

PERTURBATION OF THE EARTH IN LATITUDE.

SECTION 18.

Explanation of the method used here.

61. If φ' be the inclination of the earth's orbit to the plane of xy, and θ' the longitude of the node, then

$$\frac{d\,\theta'}{d\,t} = -\frac{n'\,a'}{\mu'\sqrt{1-e'^2}} \cdot \frac{1}{\phi'} \cdot \frac{d\,\mathrm{R}}{d\,\phi'}$$

$$\frac{d\,\phi'}{d\,t} = +\frac{n'\,a'}{\mu'\sqrt{1-e'^2}} \cdot \frac{1}{\phi'} \cdot \frac{d\,\mathrm{R}}{d\,\theta'}$$

or, neglecting e'^2,

$$\frac{d\,\theta'}{d\,t} = -\frac{n'\,a'}{\mu'} \cdot \frac{1}{\phi'} \cdot \frac{d\,\mathrm{R}}{d\,\phi'}$$

$$\frac{d\,\phi'}{d\,t} = +\frac{n'\,a'}{\mu'} \cdot \frac{1}{\phi'} \cdot \frac{d\,\mathrm{R}}{d\,\theta'}$$

These expressions are true only when ϕ' is so small that its square may be rejected. This restriction, however, is convenient as well as necessary. For in the expansion of R we shall have to proceed only to the first power of ϕ', and make $\phi' = 0$ when we have arrived at our ultimate result: consequently the same values of θ and ϕ must be employed as in the first Part.

62. The only term of R, which by expansion will produce terms of the form cos (13 − 8) with coefficients of the fifth order, is the fraction $\frac{-m}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}$. For substitution in the denominator we have

$$x' = r'\cos v' \text{ (neglecting } \phi^2)$$

$$y' = r'\sin v'$$

$$z' = r'\,.\,\phi'\,.\sin(v'-\theta')$$

$$x = r\,\{\cos(v-\theta)\,.\cos\theta - \cos\phi\,.\sin(v-\theta)\,.\sin\theta\}$$

$$y = r\,\{\cos(v-\theta)\,.\sin\theta + \cos\phi\,.\sin(v-\theta)\,.\cos\theta\}$$

$$z = r\,.\sin\phi\,.\sin(v-\theta)$$

whence the fraction is changed to

$$\frac{-m}{\sqrt{\{r'^2-2r'r.\cos(v'-v)+r^2+2r'r.f^2.\cos(v'-v)-2r'r.f^2.\cos(v'+v-2\theta)-4r'r.\phi'f.\sin(v'-\theta').\sin(v-\theta)\}}}$$

where f is put for $\sin\frac{\phi}{2}$ and $2f$ for $\sin\phi$, on the principle of (13). The part of this depending on the first power of ϕ' is

$$\frac{-m\,.\,2\,r'\,r\,.\,\phi' f\,.\sin(v'-\theta')\,.\sin(v-\theta)}{\{r'^2-2\,r'r\,.\cos(v'-v)+r^2+2\,r'r\,.f^2\,.\cos(v'-v)-2\,r'r\,.f^2\,.\cos(v'+v-2\,\theta\}^{\frac{3}{2}}}$$

of which, on the principle of (8), &c., we are to take only

$$\frac{m\,.\,r'r\,.\,\phi'f\,.\cos(v'+v-\theta'-\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2-2r'r\,.f^2\,.\cos(v'+v-2\theta)\}^{\frac{3}{2}}}$$

Expanding the denominator by powers of f^2, this becomes

$$\frac{m\,.\,r'r\,.\,\phi'f\,.\cos(v'+v-\theta'-\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{3}{2}}}+\frac{m\,.\,3r'^2r^2\,.\,\phi'f^3\,.\cos(v'+v-\theta'-\theta)\,.\cos(v'+v-2\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}$$

or

$$\frac{m\,.\,r'r\,.\,\phi'f\,.\cos(v'+v-\theta'-\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{3}{2}}}+\frac{3}{2}\,.\,\frac{m\,.\,r'^2r^2\,.\,\phi'f^3\,.\cos(2v'+2v-\theta'-3\theta)}{\{r'^2-2r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}$$

SECTION 19.

Selection of the coefficients of cos (13 – 8) *in the development of the two last fractions.*

63. If we compare the first fraction with the fraction developed in Section 8, we perceive that the following are the only differences between them. The signs of the coefficients are different: and in the coefficient of the new fraction (and in every term of its development) there is ϕ' instead of f, with the corresponding change of argument. From this it is readily seen that the coefficient of cos (13 – 8) will be formed from that in Section 8 (Art. 24), by changing the sign and multiplying by $\frac{\phi'}{f}$; the argument always being changed according to the rules of (9). The coefficient is therefore

$$-\overset{(9)}{\mathrm{M}}\,.\,e'^3\,\phi'f-\overset{(10)}{\mathrm{M}}\,.\,e'^2e\,\phi'f-\overset{(11)}{\mathrm{M}}\,.\,e'e^2\,\phi'f-\overset{(12)}{\mathrm{M}}\,.\,e^3\,\phi'f.$$

64. If we compare the second fraction with the fraction developed in Section 10, we see that there are the same differences as those mentioned above, with this additional one, that the multiplier is double of the multiplier of the fraction in Section 10. Thus the coefficient of cos (13 – 8) is found to be

$$-2\overset{(10)}{\mathrm{N}}\,.\,e'\phi'f^3-2\overset{(11)}{\mathrm{N}}\,.\,e\phi'f^3$$

The sum of the terms in these two sets, multiplied respectively by the cosines of their proper arguments, constitutes the whole term of R which we have to consider.

SECTION 20.

Numerical calculation of the perturbation in latitude.

65. The first of the terms found in the last section is $-\overset{(9)}{\mathrm{M}} . e'^3 \varphi' f . \cos \{13 (n't + \varepsilon') - 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\}$. With respect to this term only, $\frac{d\mathrm{R}}{d\varphi'} = -\overset{(9)}{\mathrm{M}} . e'^3 f . \cos \{13 (n't + \varepsilon') - 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\}$; whence

$$\theta' = \Theta' - \frac{n'a'}{\mu'\phi'}\int_t \frac{d\mathrm{R}}{d\phi'} = \Theta' + \frac{\overset{(9)}{\mathrm{M}} a'}{\mu'} . \frac{n'}{13n' - 8n} . \frac{e'^3 f}{\phi'} \sin \{13 (n't + \varepsilon')$$
$$- 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\}.$$

And

$$\frac{d\mathrm{R}}{d\theta'} = -\overset{(9)}{\mathrm{M}} . e'^3 \varphi' f . \sin \{13 (n't + \varepsilon) - 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\};$$

whence

$$\varphi' = \Phi' + \frac{n'a'}{\mu'\phi'}\int_t \frac{d\mathrm{R}}{d\theta'} = \Phi' + \frac{\overset{(9)}{\mathrm{M}} a'}{\mu'} . \frac{n'}{13n' - 8n} e'^3 f . \cos \{13 (n't + \varepsilon')$$
$$- 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\}.$$

The Earth's latitude, neglecting small terms, is $\varphi' . \sin (n't + \varepsilon' - \theta')$. And from the expression above, $\sin (n't + \varepsilon' - \theta') =$

$$\sin (n't + \varepsilon' - \Theta') - \frac{\overset{(9)}{\mathrm{M}} a'}{\mu'} . \frac{n'}{13n' - 8n} . \frac{e'^3 f}{\phi'} . \cos (n't + \varepsilon' - \Theta') . \sin$$
$$\{13 (n't + \varepsilon') - 8 (nt + \varepsilon) - 3 \varpi' - \theta' - \theta\}$$

Multiplying this by the expression for φ', and putting θ', φ', for Θ', Φ', in the small terms, we find for the latitude

$$\Phi' . \sin (n't + \varepsilon' - \Theta') - \frac{\overset{(9)}{\mathrm{M}} a'}{\mu} . \frac{n'}{13n' - 8n} . e'^3 f . \sin \{12 (n't + \varepsilon')$$
$$- 8 (nt + \varepsilon) - 3 \varpi' - \theta\}$$

and the last part, or the perturbation in latitude, is

$$- \frac{n'}{13n' - 8n} . e'^3 f . \frac{\overset{(9)}{\mathrm{M}} a'}{\mu} . \sin \{12 (n't + \varepsilon') - 8 (nt + \varepsilon) - 3 \varpi' - \theta\}$$

Similar expressions will be obtained from all the other terms.

66. If we put for $\sin\{12(n't+\epsilon')-8(nt+\epsilon)-3\varpi'-\theta\}$ its equivalent $\cos(3\varpi'+2\theta).\sin\{12(n't+\epsilon')-8(nt+\epsilon)+\theta\}-\sin(3\varpi'+2\theta).\cos\{12(n't+\epsilon')-8(nt+\epsilon)+\theta\}$, and similarly for the other terms, we find for the whole coefficient of $\sin\{12(n't+\epsilon')-8(nt+\epsilon)+\theta)\}$,

$$-\frac{n'}{13n'-8n}\cdot\frac{m}{\mu}\cdot\frac{1}{f}\cdot\frac{a'}{m}\left\{e'^3f^2.\mathrm{M}^{(9)}.\cos(3\varpi'+2\theta)\right.$$
$$+e'^2ef^2.\mathrm{M}^{(10)}.\cos(2\varpi'+\varpi+2\theta)+e'e^2f^2.\mathrm{M}^{(11)}.\cos(\varpi'+2\varpi+2\theta)$$
$$+e^3f^2.\mathrm{M}^{(12)}.\cos(3\varpi+2\theta)+2e'f^4.\mathrm{N}^{(10)}.\cos(\varpi'+4\theta)$$
$$\left.+2ef^4.\mathrm{N}^{(11)}.\cos(\varpi+4\theta)\right\}$$

and for the whole coefficient of $\cos\{12(n't+\epsilon')-8(nt+\epsilon)+\theta\}$,

$$+\frac{n'}{13n'-8n}\cdot\frac{m}{\mu}\cdot\frac{1}{f}\cdot\frac{a'}{m}\left\{e'^3f^2.\mathrm{M}^{(9)}.\sin(3\varpi'+2\theta)\right.$$
$$+e'^2ef^2.\mathrm{M}^{(10)}.\sin(2\varpi'+\varpi+2\theta)+e'e^2f^2.\mathrm{M}^{(11)}.\sin(\varpi'+2\varpi+2\theta)$$
$$+e^3f^2.\mathrm{M}^{(12)}.\sin(3\varpi+2\theta)+2e'f^4.\mathrm{N}^{(10)}.\sin(\varpi'+4\theta)$$
$$\left.+2ef^4.\mathrm{N}^{(11)}.\sin(\varpi+4\theta)\right\}$$

On performing the calculations, the inequality is found to be

$$+0'',0086.\sin\{8(nt+\epsilon)-12(n't+\epsilon')-\theta\}$$
$$+0'',0060.\cos\{8(nt+\epsilon)-12(n't+\epsilon')-\theta\}$$

or $$+0'',0105.\sin\{8(nt+\epsilon)-12(n't+\epsilon')-39^\circ 29'\}$$

which is too small to be sensible in any observations.

PART III.

PERTURBATIONS OF VENUS DEPENDING ON THE SAME ARGUMENTS.

67. If we consider Venus as disturbed by the Earth, and take the orbit of Venus for the plane xy, the term involving $\cos(13-8)$ in the expansion of R

will be exactly the same as when we consider the Earth disturbed by Venus. For the longitudes of perihelia and the longitude of the node will be the same: the sign of f will be different, but as the even powers only of this quantity enter into the expansion of R, and as its magnitude (without respect to its sign) is the same in both, that circumstance makes no difference. It is only necessary then to put m' instead of m in the multiplier of the term.

68. First, then, for the inequality in the epoch. Observing that in the expression of (51) the multiplier m is included in P, p, Q, and q, it will be seen that for the perturbation of Venus we must use the multiplier $-\frac{24\,n^2\,a\,m'}{\mu}$ instead of $\frac{39\,n'^2\,a'\,m}{\mu'}$. That is, the argument of the perturbation of Venus is the same as that of the Earth; and its coefficient is found by multiplying the coefficient for the Earth by $-\frac{8\,n^2\,a\,m'}{13\,n'^2\,a'\,m}$. As $8\,n = 13\,n'$ nearly, this fraction $= -\frac{n}{n'} \cdot \frac{a}{a'} \cdot \frac{m'}{m} = -\frac{13}{8} \cdot \frac{a}{a'} \cdot \frac{m'}{m}$ nearly. Assuming $\frac{m'}{\mu} = \frac{1}{329630}$, and the other quantities as before, this fraction is $-\frac{13}{8} \times 0{,}72333 \times \frac{401211}{329630}$: whence the long inequality in the epoch of Venus =

$$\{-2'',946 + \mathrm{Y} \times 0'',0002970\} \times \sin\{8\,l - 13\,l' + 40^\circ\ 44'\ 34'' + \mathrm{Y} \times 239'',7\}$$

$$= \{2'',946 - \mathrm{Y} \times 0'',0002970\} \times \sin\{8\,l - 13\,l' + 220^\circ\ 44'\ 34'' + \mathrm{Y} \times 239'',7\}$$

The corresponding inequality in the axis major, like that for the Earth, is insensible.

69. For the long inequality in the longitude of perihelion. This cannot be deduced from that of the Earth: but, calculating it independently in the same manner, it is found that

$$\varpi = \Pi - 0'',008 \sin(8\,l - 13\,l') - 5'',704 \,.\, \cos(8\,l - 13\,l')$$

70. For the long inequality in the excentricity. This may be derived from that in the longitude of perihelion in the same manner in which it was done for the Earth: thus it appears that

$$e = \mathrm{E} - 0{,}0000001904 \,.\, \sin(8\,l - 13\,l') + 0{,}0000000003 \,.\, \cos(8\,l - 13\,l')$$

71. For the inequality in latitude. The orbit of Venus must now be sup-

posed to be inclined at a small angle to the plane of $x\,y$. We have remarked that, in the development of R for Venus as the disturbed body, the sign of f will be changed: and as the term of R on which the perturbation in latitude depends is a multiple of odd powers of f, the sign for Venus will be different from that for the Earth. Besides this there will be no difference, except that $a\,m'\,n$ is to be substituted for $a'\,m\,n'$. Proceeding then as in (65), and considering the effect of the first term of (63), we find

$$\theta = \Theta - \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\overset{(9)}{\mathrm{M}}\,a'}{m} \cdot \frac{e'^3 f}{\varphi} . \sin\{13\,(n't + \epsilon') \\ - 8\,(n\,t + \epsilon) - 3\,\varpi' - \theta' - \theta\}$$

whence

$$\sin(nt + \epsilon - \theta) = \sin(n\,t + \epsilon - \Theta) + \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\overset{(9)}{\mathrm{M}}\,a'}{m} \cdot \frac{e'^3 f}{\Phi} \times \\ \cos(n\,t + \epsilon - \theta) . \sin\{13\,(n'\,t + \epsilon') - 8\,(n\,t + \epsilon) - 3\,\varpi' - \theta' - \theta\}$$

And

$$\varphi = \Phi - \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\overset{(9)}{\mathrm{M}}\,a'}{m} . e'^3 f . \cos\{13\,(n'\,t + \epsilon') \\ - 8\,(n\,t + \epsilon) - 3\,\varpi' - \theta' - \theta\}$$

The product of these expressions gives for the latitude of Venus

$$\Phi . \sin(n\,t + \epsilon - \Theta) + \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\overset{(9)}{\mathrm{M}}\,a'}{m} . e'^3 f . \sin\{13\,(n'\,t + \epsilon') \\ - 9\,(n\,t + \epsilon) - 3\,\varpi' - \theta'\}$$

where θ' has the same value which θ had in the investigation for the Earth. The perturbation in latitude is therefore $\frac{n}{13n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\overset{(9)}{\mathrm{M}}\,a'}{m}\, e'^3 f . \sin \{13\,(n'\,t + \epsilon') - 9\,(n\,t + \epsilon) - 3\,\varpi' - \theta'\}$, and similarly for the other terms. Comparing this with the term in (65) it will readily be seen that we have only to multiply the expression of (66) by $-\frac{m'\,n\,a}{m\,n'\,a'}$, and to put $9\,(n\,t + \epsilon) - 13\,(n'\,t + \epsilon')$ instead of $8\,(n\,t + \epsilon) - 12\,(n'\,t + \epsilon')$, and the perturbation of Venus in latitude will be found. Thus it becomes

$$-0'',0123 \,.\, \sin\{9(nt+\varepsilon)-13(n't+\varepsilon')-\theta\}-0'',0086 \,.\, \cos\{9(nt+\varepsilon)-13(n't+\varepsilon')-\theta\}$$

or

$$+0'',0151 \times \sin\{9(nt+\varepsilon)-13(n't+\varepsilon')+140^\circ\,31'\}$$

which, though larger than the Earth's perturbation in latitude, is too small to be observable.

CONCLUSION.

It appears, then, that in calculating the Earth's longitude (or 180° + Sun's longitude), the following terms should be used in addition to those that have hitherto been applied; (where l and l' are the mean tropical longitudes of Venus and the Earth, and Y the number of years after 1750:)

To the epoch of mean longitude

$$+\{2'',059-\mathrm{Y}\times 0'',0002076\}\times\sin\{8\,l-13\,l'+40^\circ\,44'\,34''+\mathrm{Y}\times 239'',7\}$$

To the epoch of longitude of perihelion

$$+2'',268\times\sin\{8\,l-13\,l'+60^\circ\,16'\}$$

To the excentricity

$$-0,0000001849\,.\,\cos\{8\,l-13\,l'+60^\circ\,16'\}$$

and that, in calculating the Earth's latitude (or the Sun's latitude with sign changed), the following term should be used;

$$+0'',0105\,.\,\sin\{8\,l-12\,l'-39^\circ\,29'\}$$

Similarly, it appears that in calculating the place of Venus, the following terms should be applied:

To the epoch of mean longitude

$$+\{2'',946-\mathrm{Y}\times 0'',0002970\}\times\sin\{8\,l-13\,l'+220^\circ\,44'\,34''+\mathrm{Y}\times 239'',7\}$$

To the longitude of perihelion

$$-5'',70\,.\,\cos\{8l-13l'\}$$

To the excentricity

$$-0,000000190\,.\,\sin\{8\,l-13\,l'\}$$

To the latitude

$$+0'',0151\,.\,\sin\{9\,l-13\,l'+140^\circ\,31'\}$$

The terms affecting the latitude may be at once neglected. The inequalities in longitude produced by the change of mean anomaly and excentricity, ($nt+\epsilon-\varpi$ and e), and which are

for the Earth

$$-0'',0470 \times \sin\{8\,l-12\,l'-15^\circ\,34'\}-0'',0346\,.\,\sin\{14\,l'-8\,l-139^\circ\,22'\}$$

for Venus

$$+0'',0671\,.\,\sin\{9\,l-13\,l'-24^\circ\,40'\}+0'',0203\,.\,\sin\{13\,l'-7\,l-168^\circ\,40'\}$$

can scarcely be detected from observation. The inequalities in the radii vectores are not sensible.

The long inequalities in the epoch of longitude are however by no means to be neglected. To point out a single instance in which their importance will be sensible, I will estimate roughly their effect on the places of the Earth and Venus at the next transit of Venus over the Sun's disk (in 1874). The value of these inequalities at the time of Bradley's observations was small; and they were at their maximum at the beginning of this century. If, then, the mean motions of the Earth and Venus were determined by comparing the observations about Bradley's time with the observations a few years ago; the Earth's longitude in 1874, when the inequalities are nearly vanishing, would be too small by nearly 4″; that of Venus would be too great by 6″: their difference of longitude would therefore be nearly 10″ in error; and this would produce on the geocentric* longitude of Venus an effect of between 20″ and 30″. As another instance, I may mention that the secular motions of the Earth, determined from observations of two consecutive centuries, would differ nearly 8″, and those of Venus nearly 12″.

These inequalities vanish in the years 1622, 1742, and 1861; and have their greatest values, positive for the Earth and negative for Venus, in 1682; and negative for the Earth and positive for Venus, in 1802. At the principal transits of Venus their values are as follows:

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* In the Memoirs of the Astronomical Society I have pointed out the utility of observations of Venus near inferior conjunction for determining the coefficient of the inequality in the Earth's motion, produced by the Moon. I take this opportunity of repeating my conviction, that observations of Venus near inferior conjunction are adapted better than any others to the detection and measurement of minute inequalities in the Earth's motion.

	For the Earth.	For Venus.
In 1639	+0″,89	−1″,28
1761	−0 ,98	+1 ,41
1769	−1 ,34	+1 ,91
1874	+0 ,68	−0 ,97
1882	+1 ,07	−1 ,53

I shall now show the coincidence of the theoretical results with the observations that first suggested their necessity.

From Burckhardt's examination of Maskelyne's observations (Connaissance des Temps, 1816), and from my examination of Mr. Pond's observations (Phil. Trans. 1828), it appeared that the mean longitudes of Delambre's tables ought to be increased

in 1783 by 0″,25
in 1801 by 0 ,08
in 1821 by 2 ,05

These observations are all reduced by the same catalogue. The differences of the corrections are not proportional to the intervals; and this is the circumstance that shows the existence of some periodical inequality.

Now the values of the argument of the long inequality in the epoch are

for 1783	240° 59′
for 1801	268 4
for the middle of 1821	298 46

The sines of these angles are −0,8745, −0,9994, −0,8766; and hence the values of the inequality were

in 1783	−1″,80
in 1801	−2 ,06
in 1821	−1 ,81

If these had been applied in the tables, the corrections given by the observations above would have been

in 1783	2″,05
in 1801	2 ,86
in 1821	3 ,86

and the differences between these are almost exactly proportional to the times. They show that the secular motion ought to be increased by $4'',8$ (the precession being supposed the same as in the application of MASKELYNE's catalogue); and then the application of the inequality investigated in this memoir will give correctly the Sun's mean longitude.

It appears, however, that the inequality in the motion of the perihelion given by this investigation, will not account for the anomalies in the place of the perihelion given in my paper referred to above.

Thus terminates one of the most laborious investigations that has yet been made in the Planetary Theory. The term in question is a striking instance of the importance to which terms, apparently the most insignificant, may sometimes rise; and the following remark will show the magnitude of the errors which might, under other circumstances, have arisen from the neglect of this term. If the perihelia of Venus and the Earth had opposite longitudes, and if the line of nodes coincided with the major axes, the excentricities and inclination having the same values as at present, the coefficient of the inequality in the epoch would be $8'',9$; and all the other terms would be important. A very small increase of the excentricities and inclination would double or treble these inequalities.

I have avoided any discussion of physical theory, as little can be added at present to what has been done by LAPLACE and others. I may remark, however, that my expression for $\frac{d\epsilon}{dt}$ differs from that given by LAPLACE; and that the difference produces no effect in the ultimate result, because LAPLACE uses $\int n\,dt$ where I have used nt. On this point I have only to state that, by adopting the expression which I have used, every formula for the longitude, the radius vector, and the velocity in any direction, is exactly the same in form for the variable ellipse as for an invariable ellipse (taking the variable elements instead of constant ones). If the disturbing force should at any instant cease, my value of ϵ for that instant would be the true value of the epoch of mean longitude in the orbit which the planet would proceed to describe. It is precisely the object of using the method of variation of elements, to obtain expressions which possess these properties; and therefore I have little doubt that my form will be recognised as more completely in accordance with the principles of that method than LAPLACE's. I should not in the present instance have raised a question on this point, but that I conceive the method of variation of elements, or some similar method, possessing the same advantages of simplicity of application and unlimited accuracy as to the order of the disturbing force, will ere long be adopted in the Planetary Theories, to the total exclusion of other methods. With this expectation, it appears important to adhere closely to the principles of the theory in every formula that is derived from it.

I believe that the paper now presented to the Royal Society contains the first[*] specific improvement in the Solar Tables made in this country since the establishment of the Theory of Gravitation. And I have great pleasure in reflecting that, after having announced a difficulty detected by observation, I have been able to offer an explanation on the grounds of physical theory.

POSTSCRIPT.

In estimating the variation of the elements of the orbit of Venus, the change of longitude of perihelion was supposed to be the same as the sidereal motion of the perihelion. This is not strictly true; as the longitude of the perihelion, measured as in Art. 4, depends upon the place of the node, and is affected therefore by the motion of the node as well as by the motion of the perihelion. The amount of the error is however perfectly insignificant.

G. B. AIRY.

Observatory, Cambridge,
Nov. 8, 1831.

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[*] I am not aware that anything has been added to the theory of planetary perturbation, by an Englishman, from the publication of NEWTON's Principia to the communication of Mr. LUBBOCK's Researches. In MASKELYNE's tables are two for the perturbations of the Earth produced by Venus and Jupiter, calculated (he states) by himself; but they are utterly useless and erroneous, as they contain no terms depending on the excentricities.

V. *Experimental Researches in Electricity.* By Michael Faraday, *F.R.S., M.R.I., Corr. Mem. Royal Acad. of Sciences of Paris, Petersburgh, &c. &c.*

Read November 24, 1831.

§ 1. *On the Induction of Electric Currents.* § 2. *On the Evolution of Electricity from Magnetism.* § 3. *On a new Electrical Condition of Matter.* § 4. *On* Arago's *Magnetic Phenomena.*

1. THE power which electricity of tension possesses of causing an opposite electrical state in its vicinity has been expressed by the general term Induction; which, as it has been received into scientific language, may also, with propriety, be used in the same general sense to express the power which electrical currents may possess of inducing any particular state upon matter in their immediate neighbourhood, otherwise indifferent. It is with this meaning that I purpose using it in the present paper.

2. Certain effects of the induction of electrical currents have already been recognised and described: as those of magnetization; Ampere's experiments of bringing a copper disc near to a flat spiral; his repetition with electro-magnets of Arago's extraordinary experiments, and perhaps a few others. Still it appeared unlikely that these could be all the effects induction by currents could produce; especially as, upon dispensing with iron, almost the whole of them disappear, whilst yet an infinity of bodies, exhibiting definite phenomena of induction with electricity of tension, still remain to be acted upon by the induction of electricity in motion.

3. Further: Whether Ampere's beautiful theory were adopted, or any other, or whatever reservation were mentally made, still it appeared very extraordinary, that as every electric current was accompanied by a corresponding intensity of magnetic action at right angles to the current, good conductors of electricity, when placed within the sphere of this action, should not have any

current induced through them, or some sensible effect produced equivalent in force to such a current.

4. These considerations, with their consequence, the hope of obtaining electricity from ordinary magnetism, have stimulated me at various times to investigate experimentally the inductive effect of electric currents. I lately arrived at positive results; and not only had my hopes fulfilled, but obtained a key which appeared to me to open out a full explanation of Arago's magnetic phenomena, and also to discover a new state, which may probably have great influence in some of the most important effects of electric currents.

5. These results I purpose describing, not as they were obtained, but in such a manner as to give the most concise view of the whole.

§. 1. *Induction of Electric Currents.*

6. About twenty-six feet of copper wire one twentieth of an inch in diameter were wound round a cylinder of wood as a helix, the different spires of which were prevented from touching by a thin interposed twine. This helix was covered with calico, and then a second wire applied in the same manner. In this way twelve helices were superposed, each containing an average length of wire of twenty-seven feet, and all in the same direction. The first, third, fifth, seventh, ninth, and eleventh of these helices were connected at their extremities end to end, so as to form one helix; the others were connected in a similar manner; and thus two principal helices were produced, closely interposed, having the same direction, not touching anywhere, and each containing one hundred and fifty-five feet in length of wire.

7. One of these helices was connected with a galvanometer, the other with a voltaic battery of ten pairs of plates four inches square, with double coppers and well charged; yet not the slightest sensible deflection of the galvanometer needle could be observed.

8. A similar compound helix, consisting of six lengths of copper and six of soft iron wire, was constructed. The resulting iron helix contained two hundred and fourteen feet of wire, the resulting copper helix two hundred and eight feet; but whether the current from the trough was passed through the copper or the iron helix, no effect upon the other could be perceived at the galvanometer.

9. In these and many other similar experiments no difference in action of any kind appeared between iron and other metals.

10. Two hundred and three feet of copper wire in one length were passed round a large block of wood; other two hundred and three feet of similar wire were interposed as a spiral between the turns of the first, and metallic contact everywhere prevented by twine. One of these helices was connected with a galvanometer, and the other with a battery of one hundred pairs of plates four inches square, with double coppers, and well charged. When the contact was made, there was a sudden and very slight effect at the galvanometer, and there was also a similar slight effect when the contact with the battery was broken. But whilst the voltaic current was continuing to pass through the one helix, no galvanometrical appearances of any effect like induction upon the other helix could be perceived, although the active power of the battery was proved to be great, by its heating the whole of its own helix, and by the brilliancy of the discharge when made through charcoal.

11. Repetition of the experiments with a battery of one hundred and twenty pairs of plates produced no other effects; but it was ascertained, both at this and the former time, that the slight deflection of the needle occurring at the moment of completing the connexion, was always in one direction, and that the equally slight deflection produced when the contact was broken, was in the other direction; and also, that these effects occurred when the first helices were used (6. 8.).

12. The results which I had by this time obtained with magnets led me to believe that the battery current through one wire, did, in reality, induce a similar current through the other wire, but that it continued for an instant only, and partook more of the nature of the electrical wave passed through from the shock of a common Leyden jar than of that from a voltaic battery, and therefore might magnetise a steel needle, although it scarcely affected the galvanometer.

13. This expectation was confirmed; for on substituting a small hollow helix, formed round a glass tube, for the galvanometer, introducing a steel needle, making contact as before between the battery and the inducing wire (7. 10.), and then removing the needle before the battery contact was broken, it was found magnetised.

14. When the battery contact was first made, then an unmagnetised needle introduced into the small indicating helix, and lastly the battery contact broken, the needle was found magnetised to an equal degree apparently with the first; but the poles were of the contrary kind.

15. The same effects took place on using the large compound helices first described (6. 8.).

16. When the unmagnetised needle was put into the indicating helix, before contact of the inducing wire with the battery, and remained there until the contact was broken, it exhibited little or no magnetism; the first effect having been nearly neutralised by the second (13. 14.). The force of the induced current upon making contact was found always to exceed that of the induced current at breaking of contact; and if therefore the contact was made and broken many times in succession, whilst the needle remained in the indicating helix, it at last came out not unmagnetised, but a needle magnetised as if the induced current upon making contact had acted alone on it. This effect may be due to the accumulation (as it is called) at the poles of the unconnected pile, rendering the current upon first making contact more powerful than what it is afterwards, at the moment of breaking contact.

17. If the circuit between the helix or wire under induction and the galvanometer or indicating spiral was not rendered complete *before* the connexion between the battery and the inducing wire was completed or broken, then no effects were perceived at the galvanometer. Thus, if the battery communications were first made, and then the wire under induction connected with the indicating helix, no magnetising power was there exhibited. But still retaining the latter communications, when those with the battery were broken, a magnet was formed in the helix, but of the second kind, i. e. with poles indicating a current in the same direction to that belonging to the battery current, or to that always induced by that current in the first instance.

18. In the preceding experiments the wires were placed near to each other, and the contact of the inducing one with the battery made when the inductive effect was required; but as some particular action might be supposed to be exerted at the moments of making and breaking contact, the induction was produced in another way. Several feet of copper wire were stretched in wide zigzag forms, representing the letter W, on one surface of a broad board; a second

wire was stretched in precisely similar forms on a second board, so that when brought near the first, the wires should everywhere touch, except that a sheet of thick paper was interposed. One of these wires was connected with the galvanometer, and the other with a voltaic battery. The first wire was then moved towards the second, and as it approached, the needle was deflected. Being then removed, the needle was deflected in the opposite direction. By first making the wires approach and then recede, simultaneously with the vibrations of the needle, the latter soon became very extensive; but when the wires ceased to move from or towards each other, the galvanometer needle soon came to its usual position.

19. As the wires approximated, the induced current was in the *contrary* direction to the inducing current. As the wires receded, the induced current was in the *same* direction as the inducing current. When the wires remained stationary, there was no induced current (54.).

20. When a small voltaic arrangement was introduced into the circuit between the galvanometer (10.) and its helix or wire, so as to cause a permanent deflection of 30° or 40°, and then the battery of one hundred pairs of plates connected with the inducing wire, there was an instantaneous action as before (11.); but the galvanometer-needle immediately resumed and retained its place unaltered, notwithstanding the continued contact of the inducing wire with the trough: such was the case in whichever way the contacts were made (33.).

21. Hence it would appear that collateral currents, either in the same or in opposite directions, exert no permanent inducing power on each other, affecting their quantity or tension.

22. I could obtain no evidence by the tongue, by spark, or by heating fine wire or charcoal, of the electricity passing through the wire under induction; neither could I obtain any chemical effects, though the contacts with metallic and other solutions were made and broken alternately with those of the battery, so that the second effect of induction should not oppose or neutralize the first (13. 16.).

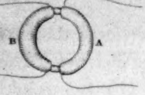
23. This deficiency of effect is not because the induced current of electricity cannot pass fluids, but probably because of its brief duration and feeble intensity; for on introducing two large copper plates into the circuit on the induced side (20.), the plates being immersed in brine, but prevented from

touching each other by an interposed cloth, the effect at the indicating galvanometer, or helix, occurred as before. The induced electricity could also pass through the trough (20.). When, however, the quantity of fluid was reduced to a drop, the galvanometer gave no indication.

24. Attempts to obtain similar effects to these by the use of wires conveying ordinary electricity were doubtful in the results. A compound helix similar to that already described (6.), and containing eight elementary helices was used. Four of the helices had their similar ends bound together by wire, and the two general terminations thus produced connected with the small magnetising helix contained an unmagnetised needle (13.). The other four helices were similarly arranged, but their ends connected with a Leyden jar. On passing the discharge, the needle was found to be a magnet; but it appeared probable that a part of the electricity of the jar had passed off to the small helix, and so magnetised the needle. There was indeed no reason to expect that the electricity of a jar possessing as it does great tension, would not diffuse itself through all the metallic matter interposed between the coatings.

25. Still it does not follow that the discharge of ordinary electricity through a wire does not produce analogous phenomena to those arising from voltaic electricity; but as it appears impossible to separate the effects produced at the moment when the discharge begins to pass, from the equal and contrary effects produced when it ceases to pass (16.), inasmuch as with ordinary electricity these periods are simultaneous, so there can be scarcely any hope that in this form of the experiment they can be perceived.

26. Hence it is evident that currents of voltaic electricity present phenomena of induction somewhat analogous to those produced by electricity of tension, although, as will be seen hereafter, many differences exist between them. The result is the production of other currents, (but which are only momentary,) parallel, or tending to parallelism, with the inducing current. By reference to the poles of the needle formed in the indicating helix (13. 14.) and to the deflections of the galvanometer-needle (11.), it was found in all cases that the induced current, produced by the first action of the inducing current, was in the contrary direction to the latter, but that the current produced by the cessation of the inducing current was in the same direction. For the purpose of avoiding periphrasis, I propose to call this action of the current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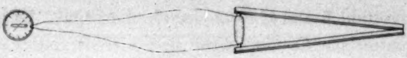
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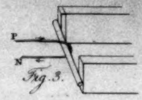
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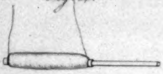
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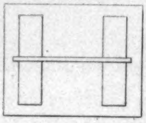
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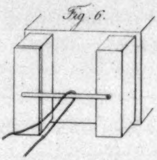
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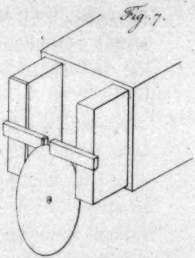
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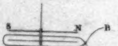
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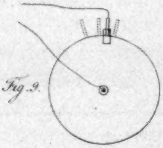
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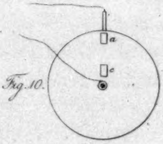
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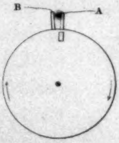
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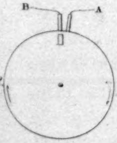
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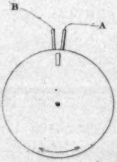
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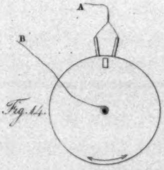
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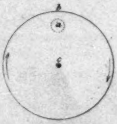
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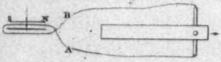
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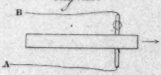


Fig. 21.

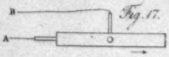


Fig. 17.

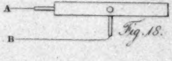
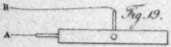


Fig. 18.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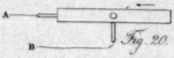
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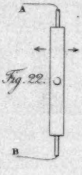
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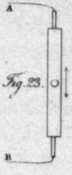
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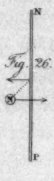
Fig. 26.

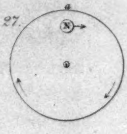
Fig. 27.

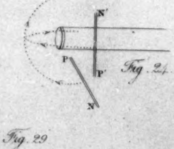
Fig. 24.

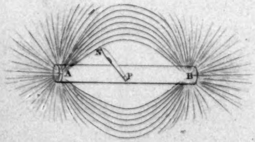
Fig. 25.

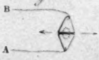
Fig. 28.

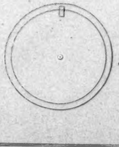
Fig. 29.

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from the voltaic battery, *volta-electric induction*. The properties of the wire, after induction has developed the first current, and whilst the electricity from the battery continues to flow through its inducing neighbour (10. 18.), constitute a peculiar electric condition, the consideration of which will be resumed hereafter. All these results have been obtained with a voltaic apparatus consisting of a single pair of plates.

§ 2. *Evolution of Electricity from Magnetism.*

27. A welded ring was made of soft round bar-iron, the metal being seven eighths of an inch in thickness, and the ring six inches in external diameter. Three helices were put round one part of this ring, each containing about twenty-four feet of copper wire one twentieth of an inch thick; they were insulated from the iron and each other, and superposed in the manner before described (6.), occupying about nine inches in length upon the ring. They could be used separately or arranged together; the group may be distinguished by the mark A (Pl. III. fig. 1.). On the other part of the ring about sixty feet of similar copper wire in two pieces were applied in the same manner, forming a helix B, which had the same common direction with the helices of A, but being separated from it at each extremity by about half an inch of the uncovered iron.

28. The helix B was connected by copper wires with a galvanometer three feet from the ring. The wires of A were connected end to end so as to form one long helix, the extremities of which were connected with a battery of ten pairs of plates four inches square. The galvanometer was immediately affected, and to a degree far beyond what has been described, when with a battery of tenfold power helices without iron were used (10.); but though the contact was continued, the effect was not permanent, for the needle soon came to rest in its natural position, as if quite indifferent to the attached electro-magnetic arrangement. Upon breaking the contact with the battery, the needle was again powerfully deflected, but in the contrary direction to that induced in the first instance.

29. Upon arranging the apparatus so that B should be out of use, the galvanometer be connected with one of the three wires of A, and the other two made into a helix through which the current from the trough (28.) was passed; similar but rather more powerful effects were produced.

30. When the battery contact was made in one direction, the galvanometer needle was deflected on the one side; if made in the other direction, the deflection was on the other side. The deflection on breaking the battery contact was always the reverse of that produced by completing it. The deflection on making a battery contact always indicated an induced current in the opposite direction to that from the battery; but on breaking the contact the deflection indicated an induced current in the same direction as that of the battery. No making or breaking of the contact at B side, or in any part of the galvanometer circuit, produced any effect at the galvanometer. No continuance of the battery current caused any deflection of the galvanometer-needle. As the above results are common to all these experiments, and to similar ones with ordinary magnets to be hereafter detailed, they need not be again particularly described.

31. Upon using the power of one hundred pair of plates (10.) with this ring, the impulse at the galvanometer, when contact was completed or broken, was so great as to make the needle spin round rapidly four or five times before the air and terrestrial magnetism could reduce its motion to mere oscillation.

32. By using charcoal at the ends of the B helix, a minute spark could be perceived when the contact of the battery with A was completed. This spark could not be due to any diversion of a part of the current of the battery through the iron to the helix B; for when the battery contact was continued, the galvanometer still resumed its perfectly indifferent state (28.). The spark was rarely seen on breaking contact. A small platina wire could not be ignited by this induced current; but there seems every reason to believe that the effect would be obtained by using a stronger original current or a more powerful arrangement of helices.

33. A feeble voltaic current was sent through the helix B and the galvanometer, so as to deflect the needle of the latter 30° or 40°, and then the battery of one hundred pairs of plates connected with A; but after the first effect was over, the galvanometer needle resumed exactly the position due to the feeble current transmitted by its own wire. This took place in whichever way the battery contacts were made, and shows that here again (20.) no permanent influence of the currents upon each other, as to their quantity and tension, exists.

34. Another arrangement was then used connecting the former experiments on volta-electric induction with the present. A combination of helices like

that already described (6.) was constructed upon a hollow cylinder of pasteboard: there were eight lengths of copper wire, containing altogether 220 feet; four of these helices were connected end to end, and then with the galvanometer (7.); the other intervening four were also connected end to end, and the battery of one hundred pairs discharged through them. In this form the effect on the galvanometer was hardly sensible (11.), but magnets could be made by the induced current (13.). But when a soft iron cylinder seven eighths of an inch thick, and twelve inches long, was introduced into the pasteboard tube, surrounded by the helices, then the induced current affected the galvanometer powerfully, and with all the phenomena just described (30.). It possessed also the power of making magnets with more energy, apparently, than when no iron cylinder was present.

35. When the iron cylinder was replaced by an equal cylinder of copper, no effect beyond that of the helices alone was produced. The iron cylinder arrangement was not so powerful as the ring arrangement already described (27.).

36. Similar effects were then produced by *ordinary magnets:* thus the hollow helix just described (34.) had all its elementary helices connected with the galvanometer by two copper wires, each five feet in length; the soft iron cylinder was introduced into its axis; a couple of bar magnets, each twenty-four inches long, were arranged with their opposite poles at one end in contact, so as to resemble a horse-shoe magnet, and then contact made between the other poles and the ends of the iron cylinder, so as to convert it for the time into a magnet (fig. 2.): by breaking the magnetic contacts, or reversing them, the magnetism of the iron cylinder could be destroyed or reversed at pleasure.

37. Upon making magnetic contact, the needle was deflected; continuing the contact, the needle became indifferent, and resumed its first position; on breaking the contact, it was again deflected, but in the opposite direction to the first effect, and then it again became indifferent. When the magnetic contacts were reversed, the deflections were reversed.

38. When the magnetic contact was made, the deflection was such as to indicate an induced current of electricity in the opposite direction to that fitted to form a magnet having the same polarity as that really produced by contact with the bar magnets. Thus when the marked and unmarked poles were placed as in fig. 3, the current in the helix was in the direction represented, P being sup-

posed to be the end of the wire going to the positive pole of the battery, or that end towards which the zinc plates face, and N the negative wire. Such a current would have converted the cylinder into a magnet of the opposite kind to that formed by contact with the poles A and B; and such a current moves in the opposite direction to the currents which in M. AMPERE's beautiful theory are considered as constituting a magnet in the position figured *.

39. But as it might be supposed that in all the preceding experiments of this section it was by some peculiar effect taking place during the formation of the magnet, and not by its mere virtual approximation, that the momentary induced current was excited, the following experiment was made. All the similar ends of the compound hollow helix (34.) were bound together by copper wire, forming two general terminations, and these were connected with the galvanometer. The soft iron cylinder (34.) was removed, and a cylindrical magnet, three quarters of an inch in diameter and eight inches and a half in length, used instead. One end of this magnet was introduced into the axis of the helix (fig. 4.), and then, the galvanometer-needle being stationary, the magnet was suddenly thrust in; immediately the needle was deflected in the same direction as if the magnet had been formed by either of the two preceding processes (34. 36.). Being left in, the needle resumed its first position, and then the magnet being withdrawn the needle was deflected in the opposite direction. These effects were not great; but by introducing and withdrawing the magnet, so that the impulse each time should be added to those previously communicated to the needle, the latter could be made to vibrate through an arc of 180° or more.

40. In this experiment the magnet must not be passed entirely through the

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* The relative position of an electric current and a magnet is by most persons found very difficult to remember, and three or four helps to the memory have been devised by M. AMPERE and others. I venture to suggest the following as a very simple and effectual assistance in these and similar latitudes. Let the experimenter think he is looking down upon a dipping needle, or upon the pole of the earth, and then let him think upon the direction of the motion of the hands of a watch, or of a screw moving direct; currents in that direction round a needle would make it into such a magnet as the dipping needle, or would themselves constitute an electro-magnet of similar qualities; or if brought near a magnet would tend to make it take that direction; or would themselves be moved into that position by a magnet so placed; or in M. AMPERE's theory are considered as moving in that direction in the magnet. These two points of the position of the dipping-needle and the motion of the watch-hands being remembered, any other relation of the current and magnet can be at once deduced from it.

helix, for then a second action occurs. When the magnet is introduced, the needle at the galvanometer is deflected in a certain direction; but being in, whether it be pushed quite through or withdrawn, the needle is deflected in a direction the reverse of that previously produced. When the magnet is passed in and through at one continuous motion, the needle moves one way, is then suddenly stopped, and finally moves the other way.

41. If such a hollow helix as that described (34.) be laid east and west (or in any other constant position), and a magnet be retained east and west, its marked pole always being one way; then whichever end of the helix the magnet goes in at, and consequently whichever pole of the magnet enters first, still the needle is deflected the same way: on the other hand, whichever direction is followed in withdrawing the magnet, the deflection is constant, but contrary to that due to its entrance.

42. These effects are simple consequences of the law hereafter to be described (114).

43. When the eight elementary helices were made one long helix, the effect was not so great as in the arrangement described. When only one of the eight helices was used, the effect was also much diminished. All care was taken to guard against any direct action of the inducing magnet upon the galvanometer, and it was found that by moving the magnet in the same direction, and to the same degree on the outside of the helix, no effect on the needle was produced.

44. The Royal Society are in possession of a large compound magnet formerly belonging to Dr. GOWIN KNIGHT, which, by permission of the President and Council, I was allowed to use in the prosecution of these experiments: it is at present in the charge of Mr. CHRISTIE, at his house at Woolwich, where, by Mr. CHRISTIE's kindness, I was at liberty to work; and I have to acknowledge my obligations to him for his assistance in all the experiments and observations made with it. This magnet is composed of about 450 bar magnets, each fifteen inches long, one inch wide, and half an inch thick, arranged in a box so as to present at one of its extremities two external poles (fig. 5.). These poles projected horizontally six inches from the box, were each twelve inches high and three inches wide. They were nine inches apart; and when a soft iron cylinder, three quarters of an inch in diameter and twelve inches long, was put across

from one to the other, it required a force of nearly one hundred pounds to break the contact. The pole to the left in the figure is the marked pole *.

45. The indicating galvanometer, in all experiments made with this magnet, was about eight feet from it, not directly in front of the poles, but about 16° or 17° on one side. It was found that on making or breaking the connexion of the poles by soft iron, the instrument was slightly affected; but all error of observation arising from this cause was easily and carefully avoided.

46. The electrical effects exhibited by this magnet were very striking. When a soft iron cylinder thirteen inches long was put through the compound hollow helix, with its ends arranged as two general terminations (39.), these connected with the galvanometer, and the iron cylinder brought in contact with the two poles of the magnet (fig. 5.), so powerful a rush of electricity took place that the needle whirled round many times in succession †.

47. Notwithstanding this great power, if the contact was continued, the needle resumed its natural position, being entirely uninfluenced by the position of the helix (30.). But on breaking the magnetic contact, the needle was whirled round in the opposite direction with a force equal to the former.

48. A piece of copper plate wrapped once round the iron cylinder like a socket, but with interposed paper to prevent contact, had its edges connected with the wires of the galvanometer. When the iron was brought in contact with the poles, the galvanometer was strongly affected.

49. Dismissing the helices and sockets, the galvanometer wire was passed over, and consequently only half round the iron cylinder (fig. 6.); but even then a strong effect upon the needle was exhibited, when the magnetic contact was made or broken.

50. As the helix with its iron cylinder was brought towards the magnetic poles, but without making contact, still powerful effects were produced. When the helix, without the iron cylinder, and consequently containing no metal but

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* To avoid any confusion as to the poles of the magnet, I shall designate the pole pointing to the north as the marked pole; I may occasionally speak of the north and south ends of the needle, but do not mean thereby north and south poles. That is by many considered the true north pole of a needle which points to the south; but in this country it is often called the south pole.

† A soft iron bar in the form of a lifter to a horse-shoe magnet, when supplied with a coil of this kind round the middle of it, becomes, by juxta-position with a magnet, a ready source of a brief but determinate current of electricity.

copper, was approached to, or placed between the poles (44.), the needle was thrown 80°, 90°, or more, from its natural position. The inductive force was of course greater, the nearer the helix, either with or without its iron cylinder, was brought to the poles; but otherwise the same effects were produced, whether the helix, &c. was or was not brought into contact with the magnet; i. e. no permanent effect on the galvanometer was produced; and the effects of approximation and removal were the reverse of each other (30.).

51. When a bolt of copper corresponding to the iron cylinder was introduced, no greater effect was produced by the helix than without it. But when a thick iron wire was substituted, the magneto-electric induction was rendered sensibly greater.

52. The direction of the electric current produced in all these experiments with the helix, was the same as that already described (38.) as obtained with the weaker bar magnets.

53. A spiral containing fourteen feet of copper wire, being connected with the galvanometer, and approximated directly towards the marked pole in the line of its axis, affected the instrument strongly; the current induced in it was in the reverse direction to the current theoretically considered by M. AMPERE as existing in the magnet (38.), or as the current in an electro-magnet of similar polarity. As the spiral was withdrawn, the induced current was reversed.

54. A similar spiral had the current of eighty pairs of 4-inch plates sent through it so as to form an electro-magnet, and then the other spiral connected with the galvanometer (53.) approximated to it; the needle vibrated, indicating a current in the galvanometer spiral the reverse of that in the battery spiral (18. 26.). On withdrawing the latter spiral, the needle passed in the opposite direction.

55. Single wires, approximated in certain directions towards the magnetic pole, had currents induced in them. On their removal, the currents were inverted. In such experiments the wires should not be removed in directions different to those in which they were approximated; for then occasionally complicated and irregular effects are produced, the causes of which will be very evident in the fourth part of this paper.

56. All attempts to obtain chemical effects by the induced current of elec-

tricity failed, though the precautions before described (22.), and all others that could be thought of, were employed. Neither was any sensation on the tongue, or any convulsive effect upon the limbs of a frog, produced. Nor could charcoal or fine wire be ignited (133.). But upon repeating the experiments more at leisure at the Royal Institution, with an armed loadstone belonging to Professor **Daniell** and capable of lifting about thirty pounds, a frog was very powerfully convulsed each time magnetic contact was made. At first the convulsions could not be obtained on breaking magnetic contact; but conceiving the deficiency of effect was because of the comparative slowness of separation, the latter act was effected by a blow, and then the frog was convulsed strongly. The more instantaneous the union or disunion is effected, the more powerful the convulsion. I thought also I could perceive the sensation upon the tongue and the flash before the eyes; but I could obtain no evidence of chemical decomposition.

57. The various experiments of this section prove, I think, most completely the production of electricity from ordinary magnetism. That its intensity should be very feeble and quantity small, cannot be considered wonderful, when it is remembered that like thermo-electricity it is evolved entirely within the substance of metals retaining all their conducting power. But an agent which is conducted along metallic wires in the manner described; which, whilst so passing possesses the peculiar magnetic actions and force of a current of electricity; which can agitate and convulse the limbs of a frog; and which, finally, can produce a spark by its discharge through charcoal (32.), can only be electricity. As all the effects can be produced by ferruginous electro-magnets (34.), there is no doubt that arrangements like the magnets of Professors **Moll**, **Henry**, **Ten Eyke**, and others, in which as many as two thousand pounds have been lifted, may be used for these experiments; in which case not only a brighter spark may be obtained, but wires also ignited, and, as the current can pass liquids (23.), chemical action be produced. These effects are still more likely to be obtained when the magneto-electric arrangements to be explained in the fourth section are excited by the powers of such apparatus.

58. The similarity of action, almost amounting to identity, between common magnets and either electro-magnets or volta-electric currents, is strikingly in accordance with and confirmatory of M. **Ampere**'s theory, and furnishes power-

ful reasons for believing that the action is the same in both cases; but, as a distinction in language is still necessary, I propose to call the agency thus exerted by ordinary magnets, *magneto-electric* or *magnelectric* induction (26.).

59. The only difference which powerfully strikes the attention as existing between volta-electric and magneto-electric induction, is the suddenness of the former and the sensible time required by the latter; but even in this early state of investigation there are circumstances which seem to indicate, that upon further inquiry this difference will, as a philosophical distinction, disappear (68.).

§. 3. *New Electrical State or Condition of Matter* *.

60. Whilst the wire is subject to either volta-electric or magneto-electric induction, it appears to be in a peculiar state; for it resists the formation of an electrical current in it, whereas, if in its common condition, such a current would be produced; and when left uninfluenced it has the power of originating a current, a power which the wire does not possess under common circumstances. This electrical condition of matter has not hitherto been recognised, but it probably exerts a very important influence in many if not most of the phenomena produced by currents of electricity. For reasons which will immediately appear (71.), I have, after advising with several learned friends, ventured to designate it as the *electro-tonic* state.

61. This peculiar condition shows no known electrical effects whilst it continues; nor have I yet been able to discover any peculiar powers exerted, or properties possessed, by matter whilst retained in this state.

62. It shows no reaction by attractive or repulsive powers. The various experiments which have been made with powerful magnets upon such metals as copper, silver, and generally those substances not magnetic, prove this point; for the substances experimented upon, if electrical conductors, must have acquired this state; and yet no evidence of attractive or repulsive powers has been observed. I have placed copper and silver discs, very delicately sus-

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* This section having been read at the Royal Society and reported upon, and having also, in consequence of a letter from myself to M. Hachette, been noticed at the French Institute, I feel bound to let it stand as part of the paper; but later investigations (intimated 73. 76. 77.) of the laws governing these phenomena, induce me to think that the latter can be fully explained without admitting the electro-tonic state. My views on this point will appear in the second series of these researches.—M. F

pended on torsion balances in vacuo, near to the poles of very powerful magnets, yet have not been able to observe the least attractive or repulsive force.

63. I have also arranged a fine slip of gold-leaf very near to a bar of copper, the two being in metallic contact by mercury at their extremities. These have been placed in vacuo, so that metal rods connected with the extremities of the arrangement should pass through the sides of the vessel into the air. I have then moved powerful magnetic poles, by this arrangement, in various directions, the metallic circuit on the outside being sometimes completed by wires, and sometimes broken. But I never could obtain any sensible motion of the gold-leaf, either directed to the magnet or towards the collateral bar of copper, which must have been, as far as induction was concerned, in a similar state to itself.

64. In some cases it has been supposed that, under such circumstances, attractive and repulsive forces have been exhibited, i. e. that such bodies have become slightly magnetic. But the phenomena now described, in conjunction with the confidence we may reasonably repose in M. Ampere's theory of magnetism, tend to throw doubt on such cases; for if magnetism depend upon the attraction of electrical currents, and if the powerful currents at first excited, both by volta-electric and magneto-electric induction, instantly and naturally cease (12. 28. 47.), causing at the same time an entire cessation of magnetic effects at the galvanometer needle, then there can be little or no expectation that any substances not partaking of the peculiar relation in which iron, nickel, and one or two other bodies, stand, should exhibit magneto-attractive powers. It seems far more probable, that the extremely feeble permanent effects observed have been due to traces of iron, or some other unrecognised cause not magnetic.

65. This peculiar condition exerts no retarding or accelerating power upon electrical currents passing through metal thus circumstanced (20. 33.). Neither could any such power upon the inducing current itself be detected; for when masses of metal, wires, helices, &c. were arranged in all possible ways by the side of a wire or helix, carrying a current measured by the galvanometer (20.), not the slightest permanent change in the indication of the instrument could be perceived. Metal in the supposed peculiar state, therefore, conducts electricity in all directions with its ordinary facility, or, in other words, its conducting power is not sensibly altered by it.

66. All metals take on the peculiar state. This is proved in the preceding experiments with copper and iron (9.), and with gold, silver, tin, lead, zinc, antimony, bismuth, mercury, &c. by experiments to be described in the fourth part (132.), admitting of easy application. With regard to iron, the experiments prove the thorough and remarkable independence of these phenomena of induction, and the ordinary magnetical appearances of that metal.

67. This state is altogether the effect of the induction exerted, and ceases as soon as the inductive force is removed. It is the same state, whether produced by the collateral passage of voltaic currents (26.), or the formation of a magnet (34. 36.), or the mere approximation of a magnet (39. 50.); and is a strong proof in addition to those advanced by M. Ampere, of the identity of the agents concerned in these several operations. It probably occurs, momentarily, during the passage of the common electric spark (24.), and may perhaps be obtained hereafter in bad conductors by weak electrical currents or other means (74. 76.).

68. The state appears to be instantly assumed (12.), requiring hardly a sensible portion of time for that purpose. The difference of time between volta-electric and magneto-electric induction, rendered evident by the galvanometer (59.), may probably be thus explained. When a voltaic current is sent through one of two parallel wires, as those of the hollow helix (34.), a current is produced in the other wire, as brief in its continuance as the time required for a single action of this kind, and which, by experiment, is found to be inappreciably small. The action will seem still more instantaneous, because, as there is an accumulation of power in the poles of the battery before contact, the first rush of electricity in the wire of communication is greater than that sustained after the contact is completed; the wire of induction becomes at the moment electro-tonic to an equivalent degree, which the moment after sinks to the state in which the continuous current can sustain it, but in sinking causes an opposite induced current to that at first produced. The consequence is, that the first induced wave of electricity more resembles that from the discharge of an electric jar, than it otherwise would do.

69. But when the iron cylinder is put into the same helix (34.), previous to the connexion being made with the battery, then the current from the latter may be considered as active in inducing innumerable currents of a similar kind

to itself in the iron, rendering it a magnet. This is known by experiment to occupy time; for a magnet so formed, even of soft iron, does not rise to its fullest intensity in an instant, and it may be because the currents within the iron are successive in their formation or arrangement. But as the magnet can induce, as well as the battery current, the combined action of the two continues to evolve induced electricity, until their joint effect is at a maximum, and thus the existence of the deflecting force is prolonged sufficiently to overcome the inertia of the galvanometer needle.

70. In all those cases where the helices or wires are advanced towards or taken from the magnet (50. 55.), the direct or inverted current of induced electricity continues for the time occupied in the advance or recession; for the electro-tonic state is rising to a higher or falling to a lower degree during that time, and the change is accompanied by its corresponding evolution of electricity; but these form no objections to the opinion that the electro-tonic state is instantly assumed.

71. This peculiar state appears to be a state of tension, and may be considered as equivalent to a current of electricity, at least equal to that produced either when the condition is induced or left at liberty. The current evolved, however, first or last, is not to be considered a measure of the degree of tension to which the electro-tonic state has risen; for as the metal retains its conducting powers unimpaired (65.), and as the electricity evolved is but for a moment, (the peculiar state being instantly assumed and lost (68.),) the electricity which may be led away by long wire conductors, offering obstruction in their substance proportionate to their small lateral and extensive linear dimensions, can be but a very small portion of that really evolved within the mass at the moment it assumes this condition. Insulated helices and portions of metal instantly assumed the state; and no traces of electricity could be discovered in them, however quickly the contact with the electrometer was made, after they were put under induction, either by the current from the battery or the magnet. A single drop of water or a small piece of moistened paper (23. 56.) was obstacle sufficient to stop the current through the conductors, the electricity evolved returning to a state of equilibrium through the metal itself, and consequently in an unobserved manner.

72. The tension of this state may therefore be comparatively very great.

But whether great or small, it is hardly conceivable that it should exist without exerting a reaction upon the original inducing current, and producing equilibrium of some kind. It might be anticipated that this would give rise to a retardation of the original current; but I have not been able to ascertain that this is the case. Neither have I in any other way as yet been able to distinguish effects attributable to such a reaction.

73. All the results favour the notion that the electro-tonic state relates to the particles, and not to the mass, of the wire or substance under induction, being in that respect different to the induction exerted by electricity of tension. If so, the state may be assumed in liquids when no electrical current is sensible, and even in non-conductors; the current itself, when it occurs, being as it were a contingency due to the existence of conducting power, and the momentary propulsive force exerted by the particles during their arrangement. Even when conducting power is equal, the currents of electricity, which as yet are the only indicators of this state, may be unequal, because of differences as to number, size, electrical condition, &c. &c. in the particles themselves. It will only be after the laws which govern this new state are ascertained, that we shall be able to predict what is the true condition of, and what are the electrical results obtainable from, any particular substance.

74. The current of electricity which induces the electro-tonic state in a neighbouring wire, probably induces that state also in its own wire; for when by a current in one wire a collateral wire is made electro-tonic, the latter state is not rendered any way incompatible or interfering with a current of electricity passing through it (62.). If, therefore, the current were sent through the second wire instead of the first, it does not seem probable that its inducing action upon the second would be less, but on the contrary more, because the distance between the agent and the matter acted upon would be very greatly diminished. A copper bolt had its extremities connected with a galvanometer, and then the poles of a battery of one hundred pairs of plates connected with the bolt, so as to send the current through it; the voltaic circuit was then suddenly broken, and the galvanometer observed for any indications of a return current through the copper bolt due to the discharge of its supposed electro-tonic state. No effect of the kind was obtained, nor indeed, for two reasons, ought it to be expected; for first, as the cessation of

induction and the discharge of the electro-tonic condition are simultaneous, and not successive, the return current would only be equivalent to the neutralization of the last portion of the inducing current, and would not therefore show any alteration of direction; or assuming that time did intervene, and that the latter current was really distinct from the former, its short, sudden character (12. 26.) would prevent it from being thus recognised.

75. No difficulty arises, I think, in considering the wire thus rendered electrotonic by its own current more than by any external current, especially when the apparent non-interference of that state with currents is considered (62. 71.). The simultaneous existence of the conducting and electro-tonic states finds an analogy in the manner in which electrical currents can be passed through magnets where it is found that both the currents passed, and those of the magnets, preserve all their properties distinct from each other, and exert their mutual actions.

76. The reason given with regard to metals extends also to fluids and all other conductors, and leads to the conclusion that when electric currents are passed through them they also assume the electro-tonic state. Should that prove to be the case, its influence in voltaic decomposition, and the transference of the elements to the poles, can hardly be doubted. In the electro-tonic state the homogeneous particles of matter appear to have assumed a regular but forced electrical arrangement in the direction of the current, which if the matter be undecomposable produces, when relieved, a return current; but in decomposable matter this forced state may be sufficient to make an elementary particle leave its companion, with which it is in a constrained condition, and associate with the neighbouring similar particle, in relation to which it is in a more natural condition, the forced electrical arrangement being itself discharged or relieved, at the same time, as effectually as if it had been freed from induction. But as the original voltaic current is continued, the electro-tonic state may be instantly renewed, producing the forced arrangement of the compound particles, to be as instantly discharged by a transference of the elementary particles of the opposite kind in opposite directions, but parallel to the current. Even the differences between common and voltaic electricity when applied to effect chemical decomposition, which Dr. Wollaston has pointed out *,

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* Philosophical Transactions, 1801. p. 247.

seem explicable by the circumstances connected with the induction of electricity from these two sources (25.). But as I have reserved this branch of the inquiry, that I might follow out the investigations contained in the present paper, I refrain (though much tempted) from offering further speculations.

77. **Marianini** has discovered and described a peculiar affection of the surfaces of metallic discs, when, being in contact with humid conductors, a current of electricity is passed through them; they are then capable of producing a reverse current of electricity, and **Marianini** has well applied the effect in explanation of the phenomena of **Ritter**'s piles[*]. M. A. **de la Rive** has described a peculiar property acquired by metallic conductors, when being immersed in a liquid as poles, they have completed, for some time, the voltaic circuit, in consequence of which, when separated from the battery and plunged in the same fluid, they themselves produce an electric current[†]. M. A. **Van Beek** has detailed cases in which the electrical relation of one metal in contact with another has been preserved after separation, and accompanied by its corresponding chemical effects[‡]. These states and results appear to differ from the electro-tonic state and its phenomena; but the true relation of the former to the latter can only be decided when our knowledge of all these phenomena has been enlarged.

78. I had occasion in the commencement of this paper (2.) to refer to an experiment by **Ampere**, as one of those dependent upon the electrical induction of currents made prior to the present investigation, and have arrived at conclusions which seem to imply doubts of the accuracy of the experiment (62, &c.): it is therefore due to M. **Ampere** that I should attend to it more distinctly. When a disc of copper (says M. **Ampere**) was suspended by a silk thread and surrounded by a helix or spiral, and when the charge of a powerful voltaic battery was sent through the spiral, a strong magnet at the same time being presented to the copper disc, the latter turned at the moment to take a position of equilibrium, exactly as the spiral itself would have turned had it been free to move. I have not been able to obtain this effect, nor indeed any motion; but the cause of my failure in the *latter* point may be due to the momentary existence of the current not allowing time for the inertia of the plate to be overcome (11. 12.). M. **Ampere** has perhaps succeeded in obtaining motion

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[*] Annales de Chimie, XXXVIII. 5. [†] Ibid. XXVIII. 190. [‡] Ibid. XXXVIII. 49.

from the superior delicacy and power of his electro-magnetical apparatus, or he may have obtained only the motion due to cessation of action. But all my results tend to invert the sense of the proposition stated by M. **Ampere**, "that a current of electricity tends to put the electricity of conductors near which it passes in motion in the same direction," for they indicate an opposite direction for the produced current (26. 53.); and they show that the effect is momentary, and that it is also produced by magnetic induction, and that certain other extraordinary effects follow thereupon.

79. The momentary existence of the phenomena of induction now described is sufficient to furnish abundant reasons for the uncertainty or failure of the experiments hitherto made to obtain electricity from magnets, or to effect chemical decomposition or arrangement by their means *.

80. It also appears capable of explaining fully the remarkable phenomena observed by M. **Arago** between metals and magnets when either are moving (120.), as well as most of the results obtained by Sir **John Herschel**, Messrs. **Babbage**, **Harris**, and others, in repeating his experiments; accounting at the same time perfectly for what at first appeared inexplicable; namely, the non-action of the same metals and magnets when at rest. These results, which also afford the readiest means of obtaining electricity from magnetism, I shall now proceed to describe.

§ 4. *Explication of* ARAGO's *Magnetic Phenomena.*

81. If a plate of copper be revolved close to a magnetic needle, or magnet,

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* The Lycée, No. 36, for January 1st, has a long and rather premature article, in which it endeavours to show anticipations by French philosophers of my researches. It however mistakes the erroneous results of MM. **Fresnel** and **Ampere** for true ones, and then imagines my true results are like those erroneous ones. I notice it here, however, for the purpose of doing honour to **Fresnel** in a much higher degree than would have been merited by a feeble anticipation of the present investigations. That great philosopher, at the same time with myself and fifty other persons, made experiments which the present paper proves could give no expected result. He was deceived for the moment, and published his imaginary success; but on more carefully repeating his trials, he could find no proof of their accuracy; and, in the high and pure philosophic desire to remove error as well as discover truth, he recanted his first statement. The example of **Berzelius** regarding the first Thorina is another instance of this fine feeling; and as occasions are not rare, it would be to the dignity of science if such examples were more frequently followed.—February 10th, 1832.

suspended in such a way that the latter may rotate in a plane parallel to that of the former, the magnet tends to follow the motion of the plate; or if the magnet be revolved, the plate tends to follow its motion; and the effect is so powerful, that magnets or plates of many pounds weight may be thus carried round. If the magnet and plate be at rest relative to each other, not the slightest effect, attractive or repulsive, or of any kind, can be observed between them (62.). This is the phenomenon discovered by M. Arago; and he states that the effect takes place not only with all metals, but with solids, liquids, and even gases, i. e. with all substances (130.).

82. Mr. Babbage and Sir John Herschel, on conjointly repeating the experiments in this country*, could obtain the effects only with the metals, and with carbon in a peculiar state (from gas retorts), i. e. only with excellent conductors of electricity. They refer the effect to magnetism induced in the plate by the magnet; the pole of the latter causing an opposite pole in the nearest part of the plate, and round this a more diffuse polarity of its own kind (120.). The essential circumstance in producing the rotation of the suspended magnet is, that the substance revolving below it shall acquire and lose its magnetism in a finite time, and not instantly (124.). This theory refers the effect to an attractive force, and is not agreed to by the discoverer, M. Arago, nor by M. Ampere, who quote against it the absence of all attraction when the magnet and metal are at rest (62. 126.), although the induced magnetism should still remain; and who, from experiments made with a long dipping needle, conceive the action to be always repulsive (125.).

83. Upon obtaining electricity from magnets by the means already described (36. 46.), I hoped to make the experiment of M. Arago a new source of electricity; and did not despair, by reference to terrestrial magneto-electric induction, of being able to construct a new electrical machine. Thus stimulated, numerous experiments were made with the magnet of the Royal Society at Mr. Christie's house, in all of which I had the advantage of his assistance. As many of these were in the course of the investigation superseded by more perfect arrangements, I shall consider myself at liberty to rearrange them in a manner calculated to convey most readily what appears to me to be a correct view of the nature of the phenomena.

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* Philosophical Transactions, 1825. p. 467.

84. The magnet has been already described (44.). To concentrate the poles, and bring them nearer to each other, two iron or steel bars, each about six or seven inches long, one inch wide, and half an inch thick, were put across the poles as in fig. 7, and being supported by twine from slipping, could be placed as near to or far from each other as was required. Occasionally two bars of soft iron were employed, so bent that when applied, one to each pole, the two smaller resulting poles were vertically over each other, either being uppermost at pleasure.

85. A disc of copper, twelve inches in diameter, and about one fifth of an inch in thickness, fixed upon a brass axis, was mounted in frames so as to be revolved either vertically or horizontally, its edge being at the same time introduced more or less between the magnetic poles (fig. 7.). The edge of the plate was well amalgamated for the purpose of obtaining a good but moveable contact; a part round the axis was also prepared in a similar manner.

86. Conductors or collectors of copper and lead were constructed so as to come in contact with the edge of the copper disc (85.), or with other forms of plates hereafter to be described (101.). These conductors were about four inches long, one third of an inch wide, and one fifth of an inch thick; one end of each was slightly grooved, to allow of more exact adaptation to the somewhat convex edge of the plates, and then amalgamated. Copper wires, one sixteenth of an inch in thickness, attached, in the ordinary manner, by convolutions to the other ends of these conductors, passed away to the galvanometer.

87. The galvanometer was roughly made, yet sufficiently delicate in its indications. The wire was of copper covered with silk, and made sixteen or eighteen convolutions. Two sewing-needles were magnetized and passed through a stem of dried grass parallel to each other, but in opposite directions, and about half an inch apart; this system was suspended by a fibre of unspun silk, so that the lower needle should be between the convolutions of the multiplier, and the upper above them. The latter was by much the most powerful magnet, and gave terrestrial direction to the whole; fig. 8. represents the direction of the wire and of the needles when the instrument was placed in the magnetic meridian; the ends of the wires are marked A and B for convenient reference hereafter. The letters S and N designate the south and north

ends of the needle when affected merely by terrestrial magnetism; the end N is therefore the marked pole (44.). The whole instrument was protected by a glass jar, and stood, as to position and distance relative to the large magnet, under the same circumstances as before (45.).

88. All these arrangements being made, the copper disc was adjusted as in fig. 7, the small magnetic poles being about half an inch apart, and the edge of the plate inserted about half their width between them. One of the galvanometer wires was passed twice or thrice loosely round the brass axis of the plate, and the other attached to a conductor (86.), which itself was retained by the hand in contact with the amalgamated edge of the disc at the part immediately between the magnetic poles. Under these circumstances all was quiescent, and the galvanometer exhibited no effect. But the instant the plate moved, the galvanometer was influenced, and by revolving the plate quickly the needle could be deflected 90° or more.

89. It was difficult under the circumstances to make the contact between the conductor and the edge of the revolving disc uniformly good and extensive; it was also difficult in the first experiments to obtain a regular velocity of rotation: both these causes tended to retain the needle in a continual state of vibration; but no difficulty existed in ascertaining to which side it was deflected, or generally, about what line it vibrated. Afterwards, when the experiments were made more carefully, a permanent deflection of the needle of nearly 45° could be sustained.

90. Here therefore was demonstrated the production of a permanent current of electricity by ordinary magnets (57.).

91. When the motion of the disc was reversed, every other circumstance remaining the same, the galvanometer needle was deflected with equal power as before; but the deflection was on the opposite side, and the current of electricity evolved, therefore, the reverse of the former.

92. When the conductor was placed on the edge of the disc a little to the right or left, as in the dotted positions fig. 9, the current of electricity was still evolved, and in the same direction as at first (88. 91.). This occurred to a considerable distance, i. e. 50° or 60° on each side of the place of the magnetic poles. The current gathered by the conductor and conveyed to the galvanometer was of the same kind on both sides of the place of greatest inten-

sity, but gradually diminished in force from that place. It appeared to be equally powerful at equal distances from the place of the magnetic poles, not being affected in that respect by the direction of the rotation. When the rotation of the disc was reversed, the direction of the current of electricity was reversed also; but the other circumstances were not affected.

93. On raising the plate, so that the magnetic poles were entirely hidden from each other by its intervention, (*a*. fig. 10,) the same effects were produced in the same order, and with equal intensity as before. On raising it still higher, so as to bring the place of the poles to *c*, still the effects were produced, and apparently with as much power as at first.

94. When the conductor was held against the edge as if fixed to it, and with it moved between the poles, even though but for a few degrees, the galvanometer needle moved and indicated a current of electricity, the same as that which would have been produced if the wheel had revolved in the same direction, the conductor remaining stationary.

95. When the galvanometer connexion with the axis was broken, and its wires made fast to two conductors, both applied to the edge of the copper disc, then currents of electricity were produced, presenting more complicated appearances, but in perfect harmony with the above results. Thus, if applied as in fig. 11, a current of electricity through the galvanometer was produced; but if their place was a little shifted, as in fig. 12, a current in the contrary direction resulted; the fact being, that in the first instance the galvanometer indicated the difference between a strong current through A and a weak one through B, and in the second, of a weak current through A and a strong one through B (92.), and therefore produced opposite deflections.

96. So also when the two conductors were equidistant from the magnetic poles, as in fig. 13, no current at the galvanometer was perceived, whichever way the disc was rotated, beyond what was momentarily produced by irregularity of contact; because equal currents in the same direction tended to pass into both. But when the two conductors were connected with one wire, and the axis with the other wire, (fig. 14,) then the galvanometer showed a current according with the direction of rotation (91.); both conductors now acting consentaneously, and as a single conductor did before (88.).

97. All these effects could be obtained when only one of the poles of the

magnet was brought near to the plate; they were of the same kind as to direction, &c., but by no means so powerful.

98. All care was taken to render these results independent of the earth's magnetism, or of the mutual magnetism of the magnet and galvanometer needles. The contacts were made in the magnetic equator of the plate, and at other parts; the plate was placed horizontally, and the poles vertically; and other precautions were taken. But the absence of any interference of the kind referred to, was readily shown by the want of all effect when the disc was removed from the poles, or the poles from the disc; every other circumstance remaining the same.

99. The relation of the current of electricity produced, to the magnetic pole, to the direction of rotation of the plate, &c. &c., may be expressed by saying, that when the unmarked pole (44. 84.) is beneath the edge of the plate, and the latter revolves horizontally, screw-fashion, the electricity which can be collected at the edge of the plate nearest to the pole is positive. As the pole of the earth may mentally be considered the unmarked pole, this relation of the rotation, the pole, and the electricity evolved, is not difficult to remember. Or if, in fig. 15, the circle represent the copper disc revolving in the direction of the arrows, and *a* the outline of the unmarked pole placed beneath the plate, then the electricity collected at *b* and the neighbouring parts is positive, whilst that collected at the centre *c* and other parts is negative (88.). The currents in the plate are therefore from the centre by the magnetic poles towards the circumference.

100. If the marked pole be placed above, all other things remaining the same, the electricity at *b*, fig. 15, is still positive. If the marked pole be placed below, or the unmarked pole above, the electricity is reversed. If the direction of revolution in any case is reversed, the electricity is also reversed.

101. It is now evident that the rotating plate is merely another form of the simpler experiment of passing a piece of metal between the magnetic poles in a rectilinear direction, and that in such cases currents of electricity are produced at right angles to the direction of the motion, and crossing it at the place of the magnetic pole or poles. This was sufficiently shown by the following simple experiment: A piece of copper plate one-fifth of an inch thick, one inch and a half wide, and twelve inches long, being amalgamated at the

edges, was placed between the magnetic poles, whilst the two conductors from the galvanometer were held in contact with its edges; it was then drawn through between the poles of the conductors in the direction of the arrow, fig. 16; immediately the galvanometer needle was deflected, its north or marked end passed eastward, indicating that the wire A received negative and the wire B positive electricity; and as the marked pole was above, the result is in perfect accordance with the effect obtained by the rotatory plate (99.).

102. On reversing the motion of the plate, the needle at the galvanometer was deflected in the opposite direction, showing an opposite current.

103. To render evident the character of the electrical current existing in various parts of the moving copper plate, differing in their relation to the inducing poles, one collector (86.) only was applied at the part to be examined near to the pole, the other being connected with the end of the plate as the most neutral place; the results are given at fig. 17—20, the marked pole being above the plate. In fig. 17, B received positive electricity; but the plate moving in the same direction, it received on the opposite side, fig. 18, negative electricity: reversing the motion of the latter, as in fig. 20, B received positive electricity; or reversing the motion of the first arrangement, that of fig. 17 to fig. 19, B received negative electricity.

104. When the plates were previously removed sideways from between the magnets, as in fig. 21, so as to be quite out of the polar axis, still the same effects were produced, though not so strongly.

105. When the magnetic poles were in contact, and the copper plate was drawn between the conductors near to the place, there was but very little effect produced. When the poles were opened by the width of a card, the effect was somewhat more, but still very small.

106. When an amalgamated copper wire, one eighth of an inch thick, was drawn through between the conductors and poles (101.), it produced a very considerable effect, though not so much as the plates.

107. If the conductors were held permanently against any particular parts of the copper plates, and carried between the magnetic poles with them, effects the same as those described were produced, in accordance with the results obtained with the revolving disc (94.).

108. On the conductors being held against the ends of the plates, and the

plates then passed between the magnetic poles, in a direction transverse to their length, the same effects were produced (fig. 22.). The parts of the plates towards the end may be considered either as mere conductors, or as portions of metal in which the electrical current is excited, according to their distance and the strength of the magnet; but the results were in perfect harmony with those before obtained. The effect was as strong as when the conductors were held against the sides of the plate (101.).

109. When the mere wire from the galvanometer, connected so as to form a complete circuit, was passed through between the poles, the galvanometer was affected; and upon passing it to and fro, so as to make the alternate impulses produced correspond with the vibrations of the needle, they could be increased to 20° or 30° on each side the magnetic meridian.

110. Upon connecting the ends of a plate of metal with the galvanometer wires, and then carrying it between the poles from end to end, (as in fig. 23.) in either direction, no effect whatever was produced upon the galvanometer. But the moment the motion became transverse, the needle was deflected.

111. These effects were also obtained from electro-magnetic poles, resulting from the use of copper helices or spirals, either alone or with iron cores (34. 54.). The directions of the motions were precisely the same; but the action was much greater when the iron cores were used, than without.

112. When a flat spiral was passed through edgeways between the poles, a curious action at the galvanometer resulted; the needle first went strongly one way, but then suddenly stopped, as if it struck against some solid obstacle, and immediately returned. If the spiral were passed through from above downwards, or from below upwards, still the motion of the needle was in the same direction, then suddenly stopped, and then was reversed. But on turning the spiral half-way round, i. e. edge for edge, then the directions of the motions were reversed, but still were suddenly interrupted and inverted as before. This double action depends upon the halves of the spiral (divided by a line passing through its centre perpendicular to the direction of its motion) acting in opposite directions; and the reason why the needle went to the same side, whether the spiral passed by the poles in the one or the other direction, depended upon the circumstance, that upon changing the motion, the direction of the wires in the approaching half of the spiral was changed also. The

effects, curious as they appear when witnessed, are immediately referable to the action of single wires (40. 109.).

113. Although the experiments with the revolving plate, wires, and plates of metal, were first successfully made with the large magnet belonging to the Royal Society, yet they were all ultimately repeated with a couple of bar magnets two feet long, one inch and a half wide, and half an inch thick; and, by rendering the galvanometer (87.) a little more delicate, with the most striking results. Ferro-electro-magnets, as those of MOLL, HENRY, &c. (57.), are very powerful. It is very essential, when making experiments on different substances, that thermo-electric effects (produced by contact of the fingers, &c.) be avoided, or at least appreciated and accounted for; they are easily distinguished by their permanency, and their independence of the magnets.

114. The relation which holds between the magnetic pole, the moving wire or metal, and the direction of the current evolved, i. e. the law which governs the evolution of electricity by magneto-electric induction, is very simple, although rather difficult to express. If in fig. 24. P N represent a horizontal wire passing by a marked magnetic pole, so that the direction of its motion shall coincide with the curved line proceeding from below upwards; or if its motion parallel to itself be in a line tangential to the curved line, but in the general direction of the arrows; or if it pass the pole in other directions, but so as to cut the magnetic curves* in the same general direction, or on the same side as they would be cut by the wire if moving along the dotted curved line;—then the current of electricity in the wire is from P to N. If it be carried in the reverse directions, the electric current will be from N to P. Or if the wire be in the vertical position, figured P′ N′, and it be carried in similar directions, coinciding with the dotted horizontal curve so far, as to cut the magnetic curves on the same side with it, the current will be from P′ to N′. If the wire be considered a tangent to the curved surface of the cylindrical magnet, and it be carried round that surface into any other position, or if the magnet itself be revolved on its axis, so as to bring any part opposite to the tangential wire,—still, if afterwards the wire be moved in the directions indi-

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* By magnetic curves, I mean the lines of magnetic forces, however modified by the juxtaposition of poles, which would be depicted by iron filings; or those to which a very small magnetic needle would form a tangent.

cated, the current of electricity will be from P to N; or if it be moved in the opposite direction, from N to P; so that as regards the motions of the wire past the pole, they may be reduced to two, directly opposite to each other, one of which produces a current from P to N, and the other from N to P.

115. The same holds true of the unmarked pole of the magnet, except that if it be substituted for the one in the figure, then, as the wires are moved in the direction of the arrows, the current of electricity would be from N to P, and as they move in the reverse direction, from P to N.

116. Hence the current of electricity which is excited in metal when moving in the neighbourhood of a magnet, depends for its direction altogether upon the relation of the metal to the resultant of magnetic action, or to the magnetic curves, and may be expressed in a popular way thus; Let A B (fig. 25.) represent a cylinder magnet, A being the marked pole, and B the unmarked pole; let P N be a silver knife-blade resting across the magnet with its edge upward, and with its marked or notched side towards the pole A; then in whatever direction or position this knife be moved edge foremost, either about the marked or the unmarked pole, the current of electricity produced will be from P to N, provided the intersected curves proceeding from A abut upon the notched surface of the knife, and those from B upon the unnotched side. Or if the knife be moved with its back foremost, the current will be from N to P in every possible position and direction, provided the intersected curves abut on the same surfaces as before. A little model is easily constructed, by using a cylinder of wood for a magnet, a flat piece for the blade, and a piece of thread connecting one end of the cylinder with the other, and passing through a hole in the blade, for the magnetic curves: this readily gives the result of any possible direction.

117. When the wire under induction is passing by an electro-magnetic pole, as for instance one end of a copper helix traversed by the electric current(34.), the direction of the current in the approaching wire is the same with that of the current in the parts or sides of the spirals nearest to it, and in the receding wire the reverse of that in the parts nearest to it.

118. All these results show that the power of inducing electric currents is circumferentially excited by a magnetic resultant or axis of power, just as circumferential magnetism is dependent upon and is exhibited by an electric current.

119. The experiments described combine to prove that when a piece of metal (and the same may be true of all conducting matter) is passed either before a single pole, or between the opposite poles of a magnet, or near electro-magnetic poles, whether ferruginous or not, electrical currents are produced across the metal transverse to the direction of motion; and which therefore, in Arago's experiments, will approximate towards the direction of radii. If a single wire be moved like the spoke of a wheel near a magnetic pole, a current of electricity is determined through it from one end towards the other. If a wheel be imagined, constructed of a great number of these radii, and this revolved near the pole, in the manner of the copper disc (85.), each radius will have a current produced in it as it passes by the pole. If the radii be supposed to be in contact laterally, a copper disc results, in which the directions of the currents will be generally the same, being modified only by the coaction which can take place between the particles, now that they are in metallic contact.

120. Now that the existence of these currents is known, Arago's phenomena may be accounted for without considering them as due to the formation in the copper of a pole of the opposite kind to that approximated, surrounded by a diffuse polarity of the same kind (82.); neither is it essential that the plate should acquire and lose its state in a finite time; nor on the other hand does it seem necessary that any repulsive force should be admitted as the cause of the rotation (82.).

121. The effect is precisely of the same kind as the electro-magnetic rotations which I had the good fortune to discover some years ago*. According to the experiments then made, which have since been abundantly confirmed, if a wire (P N, fig. 26.) be connected with the positive and negative ends of a voltaic battery, so that the positive electricity shall pass from P to N, and a marked magnetic pole N be placed near the wire between it and the spectator, the pole will move in a direction tangential to the wire, i. e. towards the right, and the wire will move tangentially towards the left, according to the directions of the arrows. This is exactly what takes place in the rotation of a plate beneath a magnetic pole; for let N (fig. 27.) be a marked pole above the circular plate, the latter being rotated in the direction of the arrow: immediately

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* Quarterly Journal of Science, vol. xii. pp. 74. 186. 416. 283.

currents of positive electricity set from the central parts in the general direction of the radii by the pole to the parts of the circumference *a* on the other side of that pole (99. 119.), and are therefore exactly in the same relation to it as the current in the wire (P N, fig. 26.) and therefore the pole in the same manner moves to the right hand.

122. If the rotation of the disc be reversed, the electric currents are reversed (91.), and the pole therefore moves to the left hand. If the contrary pole be employed, the effects are the same, i. e. in the same direction, because currents of electricity, the reverse of those described, are produced, and by reversing both poles and currents, the visible effects remain unchanged. In whatever position the magnetic axis be placed, provided the same pole be applied to the same side of the plate, the electric current produced is in the same direction, in consistency with the law already stated (114, &c.); and thus every circumstance regarding the direction of the motion may be explained.

123. These currents are discharged or return in the parts of the plate on each side of and more distant from the place of the pole, where, of course, the magnetic induction is weaker: and when the collecters are applied, and a current of electricity is carried away to the galvanometer, the deflection there is merely a repetition, by the same current or part of it, of the effect of rotation in the magnet over the plate itself.

124. It is under the point of view just put forth that I have ventured to say it is not necessary that the plate should acquire and lose its state in a finite time (120.); for if it were possible for the current to be fully developed the instant *before* it arrived at its state of nearest approximation to the vertical pole of the magnet, instead of opposite to or a little beyond it, still the relative motion of the pole and plate would be the same, the resulting force being tangential instead of direct.

125. But it is possible (though not necessary for the rotation) that time may be required for the development of the maximum current in the plate, in which case the resultant of all the forces would be in advance of the magnet when the plate is rotated, or in the rear of the magnet when the latter is rotated, and many of the effects with pure electro-magnetic poles tend to prove this is the case. Then, the tangential force may be resolved into two others, one parallel to the plane of rotation, and the other perpendicular to it; the

former would be the force excited in making the plate revolve with the magnet, or the magnet with the plate; the latter would be a repulsive force, and is probably that, the effects of which M. ARAGO has discovered (82.).

126. The extraordinary circumstance accompanying this action, which has seemed so inexplicable, namely, the cessation of all phenomena when the magnet and metal are brought to rest, now receives a full explanation (82.); for then the electrical currents which cause the motion, cease altogether.

127. All the effects of solution of metallic continuity, and the consequent diminution of power described by Messrs. BABBAGE and HERSCHEL*, now receive their natural explanation, as well also as the resumption of power when the cuts were filled up by metallic substances, which, though conductors of electricity, were themselves very deficient in the power of influencing magnets. And new modes of cutting the plate may be devised, which shall almost entirely destroy its power. Thus, if a copper plate (81.) be cut through at about a fifth or sixth of its diameter from the edge, so as to separate a ring from it, and this ring be again fastened on, but with a thickness of paper intervening (fig. 29.), and if ARAGO's experiment be made with this compound plate so adjusted that the section shall continually traverse opposite the pole, it is evident that the magnetic currents will be greatly interfered with, and the plate probably lose much of its effect†.

An elementary result of this kind was obtained by using two pieces of thick copper, shaped as in fig. 28. When the two neighbouring edges were amalgamated and put together, and the arrangement passed between the poles of the magnet, in a direction parallel to these edges, a current was urged through the wires attached to the outer angles, and the galvanometer became strongly affected; but when a single film of paper was interposed, and the experiment repeated, no sensible effect could be produced.

128. A section of this kind could not interfere much with the induction of magnetism, supposed to be of the nature ordinarily received by iron.

129. The effect of rotation or deflection of the needle, which M. ARAGO obtained by ordinary magnets, M. AMPERE succeeded in procuring by electro-

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* Philosophical Transactions, 1825, p. 481.

† This experiment has actually been made by Mr. CHRISTIE, with the results here described, and is recorded in the Philosophical Transactions for 1827. p. 82.

magnets. This is perfectly in harmony with the results relative to volta-electric and magneto-electric induction described in this paper. And by using flat spirals of copper wire, through which electric currents were sent, in place of ordinary magnetic poles (111.), sometimes applying a single one to one side of the rotating plate, and sometimes two to opposite sides, I obtained the induced currents of electricity from the plate itself, and could lead them away to, and ascertain their existence by, the galvanometer.

130. The cause which has now been assigned for the rotation in Arago's experiment, namely, the production of electrical currents, seems abundantly sufficient in all cases where the metals, or perhaps even other conductors, are concerned; but with regard to such bodies as glass, resins and, above all, gases, it seems impossible that currents of electricity capable of producing these effects should be generated in them. Yet Arago found that the effects in question were produced by these and by all bodies tried (81.). Messrs. Babbage and Herschel, it is true, did not observe them with any substance not metallic, except carbon, in a highly conducting state (82.). Mr. Harris has ascertained their occurrence with wood, marble, freestone and annealed glass, but obtained no effect with sulphuric acid and saturated solution of sulphate of iron, although these are better conductors of electricity than the former substances.

131. Future investigations will no doubt explain these difficulties, and decide the point whether the retarding or dragging action spoken of is always simultaneous with electric currents *. The existence of the action in metals, only whilst the currents exist, i. e. whilst motion is given (82. 88.), and the explication of the repulsive action observed by M. Arago (82. 125), are the strong reasons for referring it to this cause; but it may be combined with others which occasionally act alone.

132. Copper, iron, tin, zinc, lead, mercury, and all the metals tried, produced electrical currents when passed between the magnetic poles: the mercury was put into a glass tube for the purpose. The dense carbon deposited in

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* Experiments which I have since made convince me that this particular action is always due to the electrical currents formed; and they supply a test by which it may be distinguished from the action of ordinary magnetism, or any other cause, including those which are mechanical or irregular, producing similar effects.

coal gas retorts, also produced the current, but ordinary charcoal did not. Neither could I obtain any sensible effects with brine, sulphuric acid, saline solutions, &c., whether rotated in basins, or inclosed in tubes and passed between the poles.

133. I have never been able to produce any sensation upon the tongue by the wires connected with the conductors applied to the edges of the revolving plate (88.) or slips of metal (101.). Nor have I been able to heat a fine platina wire, or produce a spark, or convulse the limbs of a frog. I have failed also to produce any chemical effects by electricity thus evolved (22. 56.).

134. As the electric current in the revolving copper plate occupies but a small space, proceeding by the poles and being discharged right and left at very small distances comparatively; and as it exists in a thick mass of metal possessing almost the highest conducting power of any, and consequently offering extraordinary facility for its production and discharge; and as, notwithstanding this, considerable currents may be drawn off which can pass through narrow wires, forty, fifty, sixty, or even one hundred feet long; it is evident that the current existing in the plate itself must be a very powerful one, when the rotation is rapid and the magnet strong. This is also abundantly proved by the obedience and readiness with which a magnet ten or twelve pounds in weight follows the motion of the plate and will strongly twist up the cord by which it is suspended.

135. Two rough trials were made with the intention of constructing magneto-electric machines. In one, a ring one inch and a half broad and twelve inches external diameter, cut from a thick copper plate, was mounted so as to revolve between the poles of the magnet and represent a plate similar to those formerly used (101.), but of interminable length; the inner and outer edges were amalgamated, and the conductors applied one to each edge, at the place of the magnetic poles. The current of electricity evolved did not appear by the galvanometer to be stronger, if so strong, as that from the circular plate (88.).

136. In the other, small thick discs of copper or other metal, half an inch in diameter, were revolved rapidly near to the poles, but with the axis of rotation out of the polar axis; the electricity evolved was collected by conductors applied as before to the edges (86.). Currents were procured, but of strength much inferior to that produced by the circular plate.

137. The latter experiment is analogous to those made by Mr. Barlow with a rotating iron shell, subject to the influence of the earth *. The effects then obtained have been referred by Messrs. Babbage and Herschel to the same cause as that considered as influential in Arago's experiment†; but it would be interesting to know how far the electric current which might be produced in the experiment would account for the deflexion of the needle. The mere inversion of a copper wire six or seven times near the poles of the magnet, and isochronously with the vibrations of the galvanometer needle connected with it, was sufficient to make the needle vibrate through an arc of 60° or 70°. The rotation of a copper shell would perhaps decide the point, and might even throw light upon the more permanent, though somewhat analogous effects obtained by Mr. Christie.

138. The remark which has already been made respecting iron (66.), and the independence of the ordinary magnetical phenomena of that substance, and the phenomena now described of magneto-electric induction in that and other metals, was fully confirmed by many results of the kind detailed in this section. When an iron plate similar to the copper one formerly described (101.) was passed between the magnetic poles, it gave a current of electricity like the copper plate, but decidedly of less power; and in the experiments upon the induction of electric currents (9.), no difference in the kind of action between iron and other metals could be perceived. The power therefore of an iron plate to drag a magnet after it, or to intercept magnetic action, should be carefully distinguished from the similar power of such metals as silver, copper, &c. &c. inasmuch as in the iron by far the greater part of the effect is due to what may be called ordinary magnetic action. There can be no doubt that the cause assigned by Messrs. Babbage and Herschel in explication of Arago's phenomena is true when iron is the metal used.

139. The very feeble powers which were found by those philosophers to belong to bismuth and antimony, when moving, of affecting the suspended magnet, and which has been confirmed by Mr. Harris, seem at first disproportionate to their conducting powers; whether it be so or not must be decided by future experiment (73.) ‡. These metals are highly crystalline, and probably conduct

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* Philosophical Transactions, 1825. p. 317.

† Ibid. 1825. p. 485.

‡ I have since been able to explain these differences, and prove, with several metals, that the effect is in the order of the conducting power; for I have been able to obtain, by magneto-electric induction,

electricity with different degrees of facility in different directions; and it is not unlikely that where a mass is made up of a number of crystals heterogeneously associated, an effect approaching to that of actual division may occur (127.); or the currents of electricity may become more suddenly deflected at the confines of similar crystalline arrangements, and so be more readily and completely discharged within the mass.

currents of electricity which are proportionate in strength to the conducting power of the bodies experimented with (211.).

Royal Institution, November 1831.

Note.—In consequence of the long period which has intervened between the reading and printing of the foregoing paper, accounts of the experiments have been dispersed, and, through a letter of my own to M. Hachette, have reached France and Italy. That letter was translated (with some errors), and read to the Academy of Sciences at Paris, 26th December, 1831. A copy of it in *Le Temps* of the 28th December quickly reached Signor Nobili, who, with Signor Antinori, immediately experimented upon the subject, and obtained many of the results mentioned in my letter; others they could not obtain or understand, because of the brevity of my account. These results by Signori Nobili and Antinori have been embodied in a paper dated 31st January 1832, and printed and published in the number of the *Antologia* dated November 1831, (according at least to the copy of the paper kindly sent me by Signor Nobili). It is evident the work could not have been then printed; and though Signor Nobili, in his paper, has inserted my letter as the text of his experiments, yet the circumstance of back date has caused many here, who have heard of Nobili's experiments by report only, to imagine his results were anterior to, instead of being dependent upon, mine.

I may be allowed under these circumstances to remark, that I experimented on this subject several years ago, and have published results. (See Quarterly Journal of Science for July 1825. p. 338.) The following also is an extract from my note-book, dated November 28, 1825: "Experiments on induction by connecting wire of voltaic battery:—a battery of four troughs, ten pairs of plates, each arranged side by side—the poles connected by a wire about four feet long, parallel to which was another similar wire separated from it only by two thicknesses of paper, the ends of the latter were attached to a galvanometer:—exhihited no action, &c. &c. &c.—Could not in any way render any induction evident from the connecting wire." The cause of failure at that time is now evident (79.).—M. F. April 1832.

VI. THE BAKERIAN LECTURE.—*Experimental Researches in Electricity.—Second Series. By* MICHAEL FARADAY, *F.R.S., M.R.I., Corr. Mem. Royal Acad. of Sciences of Paris, Petersburgh, &c. &c.*

Read January 12, 1832.

§ 5. *Terrestrial Magneto-electric Induction.*

§ 6. *Force and Direction of Magneto-electric Induction generally.*

§. 5. *Terrestrial Magneto-electric Induction.*

140. WHEN the general facts described in the former paper were discovered, and the law of magneto-electric induction relative to direction was ascertained (114.), it was not difficult to perceive that the earth would produce the same effect as a magnet, and to an extent that would, perhaps, render it available in the construction of new electrical machines. The following are some of the results obtained in pursuance of this view.

141. The hollow helix already described (6.) was connected with the galvanometer by wires eight feet long; and the soft iron cylinder (34.), after being heated red hot, and slowly cooled, to remove all traces of magnetism, was put into the helix so as to project equally at both ends, and fixed there. The combined helix and bar were held in the magnetic direction or line of dip, and (the galvanometer needle being motionless) were then inverted, so that the lower end should become the upper, but the whole still correspond to the magnetic direction; the needle was immediately deflected. As it returned to its first position, the helix and bar were again inverted; and by doing this two or three times, making the inversions and vibrations to coincide, the needle swung through an arc of 150° or 160°.

142. When one end of the helix, which may be called A, was uppermost at first (B end consequently being below), then it mattered not in which direction it proceeded during the inversion, whether to the right hand or left hand, or through any other course; still the galvanometer needle passed in the same direction. Again, when B end was uppermost, the inversion of the helix and bar in any direction always caused the needle to be deflected the same way;

that way being the opposite to the course of the deflection taken in the former general case.

143. When the helix in any given position was inverted, the effect was as if a magnet with its marked pole downwards had been introduced from above into the inverted helix. Thus, if the end B were upwards, such a magnet introduced from above would make the marked end of the galvanometer needle pass west. Or the end A being upwards, and the soft iron in its place, inversion of the whole produced the same effect.

144. When the soft iron bar was taken out of the helix and inverted in various directions within four feet of the galvanometer, not the slightest effect upon it was produced.

145. These phenomena are the necessary consequence of the inductive magnetic power of the earth, rendering the soft iron cylinder a magnet with its marked pole downwards. The experiment is analogous to that in which two bar magnets were used to magnetize the same cylinder in the same helix (36.), and the inversion of position in the present experiment is equivalent to a change of the poles in that arrangement. But the result is not less an instance of the evolution of electricity by means of the magnetism of the globe.

146. The helix alone was then permanently held in the magnetic direction, and the soft iron cylinder afterwards introduced; the galvanometer needle was instantly deflected; by withdrawing the bar as the needle returned, and continuing the two actions simultaneously, the vibrations soon extended through an arc of 180°. The effect was precisely the same as that of using a cylinder magnet with its marked pole downwards; and the direction of motion, &c. was perfectly in accordance with those obtained in the former experiments with such a magnet (39.). A magnet in that position was then used, and gave the same deflections, but stronger. When the helix was put at right angles to the magnetic direction or dip, then the introduction or removal of the soft iron cylinder produced no effect at the needle. Any inclination to the dip gave results of the same kind as those already described, but increasing in strength as the helix approximated to the line of the dip.

147. The cylinder magnet, although it has great power of affecting the galvanometer when moving into or out of the helix, has no power of continuing the deflection (39.); and therefore, though left in, still the magnetic

needle comes to its usual place of rest. But upon repeating the experiment of inversion in the direction of the dip (141.), the needle was affected as powerfully as before; the disturbance of the magnetism in the steel magnet, by the earth's inductive force upon it, being thus shown to be nearly, if not quite, equal in amount and rapidity to that occurring in soft iron. It is probable that in this way magneto-electrical arrangements may become very useful in indicating the disturbance of magnetic forces, where other means will not apply; for it is not the whole magnetic power which produces the visible effect, but only the difference due to the disturbing causes.

148. These favourable results led me to hope that the direct magneto-electric induction of the earth might be rendered sensible; and I ultimately succeeded in obtaining the effect in several ways. When the helix just referred to (141. 6.) was placed in the magnetic dip, but without any cylinder of iron or steel, and was then inverted, a feeble action at the needle was observed. Inverting the helix ten or twelve times, and at such times that the deflecting forces exerted by the currents of electricity produced in it should be added to the momentum of the needle (39.), the latter was soon made to vibrate through an arc of 80° or 90°. Here, therefore, currents of electricity were produced by the direct inductive power of the earth's magnetism, without the use of any ferruginous matter, and upon a metal not capable of exhibiting any of the ordinary magnetic phenomena. The experiment in everything represents the effects produced by bringing the same helix to one or both poles of any powerful magnet (50.).

149. Guided by the law already expressed (114.), I expected that all the electric phenomena of the revolving metal plate could now be produced without any other magnet than the earth. The plate so often referred to (85.) was therefore fixed so as to rotate in a horizontal plane. The magnetic curves of the earth (114. *note*), i. e. the dip, passes through this plane at angles of about 70°, which it was expected would be an approximation to perpendicularity, quite enough to allow of magneto-electric induction sufficiently powerful to produce a current of electricity.

150. Upon rotation of the plate, the currents ought, according to the law (114. 121.), to tend to pass in the direction of the radii, through *all* parts of the plate, either from the centre to the circumference, or from the circumference to the centre, as the direction of the rotation of the plate was one way or the

other. One of the wires of the galvanometer was therefore brought in contact with the axis of the plate, and the other attached to a leaden collector or conductor (86.), which itself was placed against the amalgamated edge of the disc. On rotating the plate there was a distinct effect at the galvanometer needle; on reversing the rotation, the needle went in the opposite direction; and by making the action of the plate coincide with the vibrations of the needle, the arc through which the latter passed soon extended to half a circle.

151. Whatever part of the edge of the plate was touched by the conductor, the electricity was the same, provided the direction of rotation continued unaltered.

152. When the plate revolved *screw-fashion*, or as the hands of a watch, the current of electricity (150.) was from the centre to the circumference; when the direction of rotation was *unscrew*, the current was from the circumference to the centre. These directions are the same with those obtained when the unmarked pole of a magnet was placed beneath the revolving plate (99.).

153. When the plate was in the magnetic meridian, or in any other plane coinciding with the magnetic dip, then its rotation produced no effect upon the galvanometer. When inclined to the dip but a few degrees, electricity began to appear upon rotation. Thus when standing upright in a plane perpendicular to the magnetic meridian, and when consequently its own plane was inclined only 20° to the dip, revolution of the plate evolved electricity. As the inclination was increased, the electricity became more powerful until the angle formed by the plane of the plate with the dip was 90°, when the electricity for a given velocity of the plate was a maximum.

154. It is a striking thing to observe the revolving copper plate become thus a new electrical machine; and curious results arise on comparing it with the common machine. In the one, the plate is of the best non-conducting substance that can be applied; in the other, it is the most perfect conductor: in the one, insulation is essential; in the other, it is fatal. In comparison of the quantities of electricity produced, the metal machine does not at all fall below the glass one; for it can produce a constant current capable of deflecting the galvanometer needle, whereas the latter cannot. It is quite true that the force of the current thus evolved has not as yet been increased so as to render it available in any of our ordinary applications of this power; but there appears every reasonable expectation that this may hereafter be effected; and probably

by several arrangements. Weak as the current may seem to be, it is as strong as, if not stronger than, any thermo-electric current; for it can pass fluids (23.), agitate the animal system, and in the case of an electro-magnet has produced sparks (32.).

155. A disc of copper, one fifth of an inch thick and only one inch and a half in diameter, was amalgamated at the edge; a square piece of sheet lead, (copper would have been better) of equal thickness had a circular hole cut in it, into which the disc loosely fitted; a little mercury completed the metallic communication of the disc and its surrounding ring; the latter was attached to one of the galvanometer wires, and the other wire dipped into a little metallic cup containing mercury, fixed upon the top of the copper axis of the small disc. Upon rotating the disc in a horizontal plane, the galvanometer needle could be affected, although the earth was the only magnet employed, and the radius of the disc but three quarters of an inch; in which space only the current was excited.

156. On putting the pole of a magnet under the revolving disc, the galvanometer needle could be permanently deflected.

157. On using copper wires one sixth of an inch in thickness instead of the smaller wires (86.) hitherto constantly employed, far more powerful effects were obtained. Perhaps if the galvanometer had consisted of fewer turns of thick wire instead of many convolutions of thinner, more striking effects would have been produced.

158. One form of apparatus which I purpose having arranged, is to have several discs superposed; the discs are to be metallically connected, alternately at the edges and at the centres, by means of mercury; and are then to be revolved alternately in opposite directions, i. e. the first, third, fifth, &c. to the right hand, and the second, fourth, sixth, &c. to the left hand; the whole being placed so that the discs are perpendicular to the dip, or intersect most directly the magnetic curves of powerful magnets. The electricity will be from the centre to the circumference in one set of discs, and from the circumference to the centre in those on each side of them; thus the action of the whole will conjoin to produce one combined and more powerful current.

159. I have rather, however, been desirous of discovering new facts and new relations dependent on magneto-electric induction, than of exalting the

force of those already obtained; being assured that the latter would find their full development hereafter.

160. I referred in my former paper to the probable influence of terrestrial magneto-electric induction (137.) in producing, either altogether or in part, the phenomena observed by Messrs. CHRISTIE and BARLOW *, whilst revolving ferruginous bodies; and especially those observed by the latter when rapidly rotating an iron shell, and which were by that philosopher referred to a change in the ordinary disposition of the magnetism of the ball. I suggested also that the rotation of a copper globe would probably insulate the effects due to electric currents from those due to mere derangement of magnetism, and throw light upon the true nature of the phenomena.

161. Upon considering the law already referred to (114.), it appeared impossible that a metallic globe could revolve under natural circumstances, without having electric currents produced within it, circulating round the revolving globe in a plane at right angles to the plane of revolution, provided its axis of rotation did not coincide with the dip; and it appeared that the current would be most powerful when the axis of revolution was perpendicular to the dip of the needle: for then all those parts of the ball below a plane passing through its centre and perpendicular to the dip, would in moving cut the magnetic curves in one direction, whilst all those parts above that plane would cut them in the other direction: currents therefore would exist in these moving parts, proceeding from one pole of rotation to the other; but the currents above would be in the reverse direction to those below, and in conjunction with them would produce a continued circulation of electricity.

162. As the electric currents are nowhere interrupted in the ball, powerful effects were expected, and I endeavoured to obtain them with simple apparatus. The ball I used was of brass; it had belonged to an old electrical machine, was hollow, thin (too thin), and four inches in diameter; a brass wire was screwed into it, and the ball either turned in the hand by the wire, or sometimes, to render it more steady, supported by its wire in a notched piece of wood, and motion again given by the hand. The ball gave no signs of magnetism when at rest.

163. A compound magnetic needle was used to detect the currents. It was

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* CHRISTIE, Phil. Trans. 1825. pp. 58. 347, &c. BARLOW, Phil. Trans. 1825. p. 317.

arranged thus: a sewing-needle had the head and point broken off, and was then magnetised; being broken in half, the two magnets thus produced were stuck into a stem of dried grass, so as to be perpendicular to it, and about four inches asunder; they were both in one plane, but their similar poles in contrary directions. The grass was attached to a piece of unspun silk about six inches long, the latter to a stick passing through a cork in the mouth of a cylindrical jar; and thus a compound arrangement was obtained, perfectly sheltered from the motion of the air, but little influenced by the magnetism of the earth, and yet highly sensible to magnetic and electric forces, when the latter were brought into the vicinity of the one or the other needle.

164. Upon adjusting the needles to the plane of the magnetic meridian; arranging the ball on the outside of the glass jar to the west of the needles, and at such a height that its centre should correspond horizontally with the upper needle, whilst its axis was in the plane of the magnetic meridian, but perpendicular to the dip; and then rotating the ball, the magnet was immediately affected. Upon inverting the direction of rotation, the needle was again affected, but in the opposite direction. When the ball revolved from east over to west, the marked pole went eastward; when the ball revolved in the opposite direction, the marked pole went westward or towards the ball. Upon placing the ball to the east of the needles, still the needle was deflected in the same way; i. e. when the ball revolved from east over to west, the marked pole went eastward (or towards the ball); when the rotation was in the opposite direction, the marked pole went westward.

165. By twisting the silk of the needles, the latter were brought into a position perpendicular to the plane of the magnetic meridian; the ball was again revolved, with its axis parallel to the needle; the needle was affected as before, and the deflection was such as to show that both here and in the former case the needle was influenced solely by currents of electricity existing in the brass globe.

166. If the upper part of the revolving ball be considered as a wire moving from east to west, over the unmarked pole of the earth, the current of electricity in it should be from north to south (99. 114. 150.); if the under part be considered as a similar wire, moving from west to east over the same pole, the electric current should be from south to north; and the circulation of electri-

city should therefore be from north above to south, and below back to north, in a metal ball revolving from east above to west in these latitudes. Now these currents are exactly those required to give the directions of the needle in the experiments just described; so that the coincidence of the theory from which the experiments were deduced with the experiments themselves, is perfect.

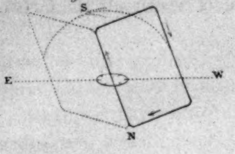
167. Upon inclining the axis of rotation considerably, the revolving ball was found to affect the magnetic needle; and it was not until the angle which it formed with the magnetic dip was rendered small, that its effects, even upon this apparatus, were lost (153.). When revolving with its axis parallel to the dip, it is evident that the globe becomes analogous to the copper plate; electricity of one kind might be collected at its equator, and of the other kind at its poles.

168. A current in the ball, such as that described above (161.), although it ought to deflect a needle the same way whether it be to the right or the left of the ball and of the axis of rotation, ought to deflect it the contrary way when above or below the ball; for then the needle is, or ought to be, acted upon in a contrary direction by the current. This expectation was fulfilled by revolving the ball beneath the magnetic needle, the latter being still inclosed in its jar. When the ball was revolved from east over to west, the marked pole of the needle, instead of passing eastward, went westward; and when revolved from west over to east, the marked pole went eastward.

169. The deflections of the magnetic needle thus obtained with a brass ball are exactly in the same direction as those observed by Mr. Barlow in the revolution of the iron shell; and from the manner in which iron exhibits the phenomena of magneto-electric induction like any other metal, and distinct from its peculiar magnetic phenomena (132.), it is impossible but that electric currents must have been excited, and become active in those experiments. What proportion of the whole effect obtained is due to this cause, must be decided by a more mature investigation of all the phenomena.

170. These results, in conjunction with the general law before described, suggested an experiment of extreme simplicity, which yet, on trial, was found to answer perfectly. The exclusion of all extraneous circumstances and all complexity of arrangement, and the distinct character of the indications

Fig: 30.



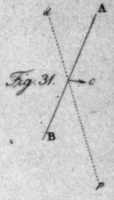
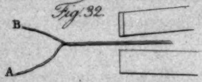


Fig: 32.



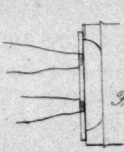


Fig: 33.

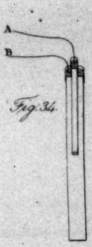


Fig: 34.

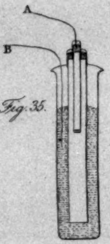


Fig: 35.

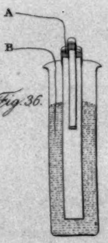
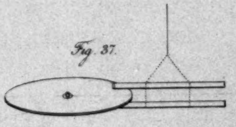


Fig: 36.

Fig. 37.



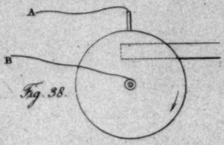
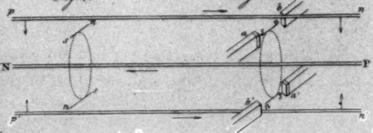


Fig: 38.

Fig: 40.

Fig: 41.



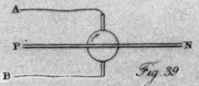


Fig: 39.

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afforded, render this single experiment an epitome of nearly all the facts of magneto-electric induction.

171. A piece of common copper wire, about eight feet long and one twentieth of an inch in thickness, had one of its ends fastened to one of the terminations of the galvanometer wire, and the other end to the other termination; thus it formed an endless continuation of the galvanometer wire: it was then roughly adjusted into the shape of a rectangle, or rather of a loop, the upper part of which could be carried to and fro over the galvanometer, whilst the lower part, and the galvanometer attached to it, remained steady (Plate IV. fig. 30.). Upon moving this loop over the galvanometer from right to left, the magnetic needle was immediately deflected; upon passing the loop back again, the needle passed in the contrary direction to what it did before; upon repeating these motions of the loop in accordance with the vibrations of the needle (39.), the latter soon swung through 90° or more.

172. The relation of the current of electricity produced in the wire, to its motion, may be understood by supposing the convolutions at the galvanometer away, and the wire arranged as a rectangle, with its lower edge horizontal and in the plane of the magnetic meridian, and a magnetic needle suspended above and over the middle part of this edge, and directed by the earth (fig. 30.). On passing the upper part of the rectangle from west to east into the position represented by the dotted line, the marked pole of the magnetic needle went west; the electric current was therefore from north to south in the part of the wire passing under the needle, and from south to north in the moving or upper part of the parallelogram. On passing the upper part of the rectangle from east to west over the galvanometer, the marked pole of the needle went east, and the current of electricity was therefore the reverse of the former.

173. When the rectangle was arranged in a plane east and west, and the magnetic needle made parallel to it, either by the torsion of its suspension thread or the action of a magnet, still the general effects were the same. On moving the upper part of the rectangle from north to south, the marked pole of the needle went north; when the wire was moved in the opposite direction, the marked pole went south. The same effect took place when the motion of the wire was in any other azimuth of the line of dip; the direction of the cur-

rent always being conformable to the law formerly expressed (114.), and also to the directions obtained with the rotating ball (164.).

174. In these experiments it is not necessary to move the galvanometer or needle from its first position. It is quite sufficient if the wire of the rectangle is distorted where it leaves the instrument, and bent so as to allow the moving upper part to travel in the desired direction.

175. The moveable part of the wire was then arranged *below* the galvanometer, but so as to be carried across the dip. It affected the instrument as before, and in the same direction; i. e. when carried from west to east under the instrument, the marked end of the needle went west, as before. This should, of course, be the case; for when the wire is cutting the magnetic dip in a certain direction, an electric current also in a certain direction should be induced in it.

176. If in fig. 31. *d p* be parallel to the dip, and B A be considered as the upper part of the rectangle (171.), with an arrow *c* attached to it, both these being retained in a plane perpendicular to the dip,—then, however B A with its attached arrow is moved upon *d p* as an axis, if it afterwards proceed in the direction of the arrow, a current of electricity will move along it from B towards A.

177. When the moving part of the wire was carried up or down parallel to the dip, no effect was produced on the galvanometer. When the direction of motion was a little inclined to the dip, electricity manifested itself; and was at a maximum when the motion was perpendicular to the magnetic direction.

178. When the wire was bent into other forms and moved, equally strong effects were obtained, especially when instead of a rectangle a double catenarian curve was formed of it on one side of the galvanometer, and the two single curves or halves were swung in opposite directions at the same time; their action then combined to affect the galvanometer: but all the results were reducible to those above described.

179. The longer the extent of the moving wire, and the greater the space through which it moves, the greater is the effect upon the galvanometer.

180. The facility with which electric currents are produced in metals when moving under the influence of magnets, suggests that henceforth precautions should always be taken, in experiments upon metals and magnets, to guard

against such effects. Considering the universality of the magnetic influence of the earth, it is a consequence which appears very extraordinary to the mind, that scarcely any piece of metal can be moved in contact with others, either at rest, or in motion with different velocities or in other directions, without an electric current existing within them. It is probable that amongst arrangements of steam-engines and metal machinery, some curious accidental magneto-electric combinations may be found, producing effects which have never been observed, or, if noticed, have never as yet been understood.

181. Upon considering the effects of terrestrial magneto-electric induction which have been described, it is almost impossible to resist the impression that similar effects, but infinitely greater in force, may be produced by the action of the magnet of the globe upon its own mass, in consequence of its diurnal rotation. It would seem that if a bar of metal be laid in these latitudes on the surface of the earth parallel to the magnetic meridian, a current of electricity tends to pass through it from south to north, in consequence of the travelling of the bar from west to east (172.), by the rotation of the earth; that if another bar in the same direction be connected with the first by wires, it cannot discharge the current of the first, because it has an equal tendency to have a current in the same direction induced within itself: but that if the latter be carried from east to west, which is equivalent to a diminution of the motion communicated to it from the earth (172.), then the electric current from south to north is rendered evident in the first bar, in consequence of its discharge, at the same time, by means of the second.

182. Upon the supposition that the rotation of the earth tended, by magneto-electric induction, to cause currents in its own mass, these would, according to the law (114.) and the experiments, be, upon the surface at least, from the parts in the neighbourhood of or towards the plane of the equator, in opposite directions to the poles; and if collectors could be applied at the equator and at the poles of the globe, as has been done with the revolving copper plate (150.), and also with magnets (220.), then negative electricity would be collected at the equator, and positive electricity at both poles (222.). But without the conductors, or something equivalent to them, it is evident these currents could not exist, as they could not be discharged.

183. I did not think it impossible that some natural difference might occur

between bodies, relative to the intensity of the current produced or tending to be produced in them by magneto-electric induction, which might be shown by opposing them to each other; especially as Messrs. ARAGO, BABBAGE, HERSCHELL, and HARRIS have all found great differences, not only between the metals and other substances, but between the metals themselves, in their power of receiving motion from or giving it to a magnet in trials by revolution (130.). I therefore took two wires, each one hundred and twenty feet long, one of iron and the other of copper. These were connected with each other at their ends, and then extended in the direction of the magnetic meridian, so as to form two nearly parallel lines, nowhere in contact except at the extremities. The copper wire was then divided in the middle, and examined by a delicate galvanometer, but no evidence of an electrical current was obtained.

184. By favour of His Royal Highness the President of the Society, I obtained the permission of HIS MAJESTY to make experiments at the lake in the gardens of Kensington-palace, for the purpose of comparing, in a similar manner, water and metal. The basin of this lake is artificial; the water is supplied by the Chelsea Company; no springs run into it, and it presented what I required, namely, a uniform mass of still pure water, with banks ranging nearly from east to west, and from north to south.

185. Two perfectly clean bright copper plates, each exposing four square feet of surface, were soldered to the extremities of a copper wire; the plates were immersed in the water, north and south of each other, the wire which connected them being arranged upon the grass of the bank. The plates were about four hundred and eighty feet from each other, in a right line; the wire was probably six hundred feet long. This wire was then divided in the middle, and connected by two cups of mercury with a delicate galvanometer.

186. At first, indications of electric currents were obtained; but when these were tested by inverting the direction of contact, and in other ways, they were found to be due to other causes than the one sought for. A little difference in temperature; a minute portion of the nitrate of mercury used to amalgamate the wires, entering into the water employed to reduce the two cups of mercury to the same temperature; was sufficient to produce currents of electricity, which affected the galvanometer, notwithstanding they had to pass nearly five hundred feet of water. When these and other interfering causes were guarded

against, no effect was obtained; and it appeared that even such dissimilar substances as water and copper, when cutting the magnetic curves of the earth with equal velocity, perfectly neutralized each other's action.

187. Mr. Fox of Falmouth has obtained some highly important results respecting the electricity of metalliferous veins in the mines of Cornwall, which have been published in the Philosophical Transactions*. I have examined the paper with a view to ascertain whether any of the effects were probably referable to magneto-electric induction; but, though unable to form a very strong opinion, believe they are not. When parallel veins running east and west were compared, the general tendency of the electricity *in the wires* was from north to south; when the comparison was made between parts towards the surface and at some depth, the current of electricity in the wires was from above downwards. If there should be any natural difference in the force of the electric currents produced by magneto-electric induction in different substances, or substances in different positions moving with the earth, and which might be rendered evident by increasing the masses acted upon, then the wires and veins experimented with by Mr. Fox might perhaps have acted as dischargers to the electricity of the mass of strata included between them, and the directions of the currents would be those observed as above.

188. Although the electricity obtained by magneto-electric induction in a few feet of wire is of but small intensity, and has not as yet been observed except in metals, and carbon in a particular state, still it has power to pass through brine (23.); and, as increased length in the substance acted upon produces increase of intensity, I hoped to obtain effects from extensive moving masses of water, though still water gave none. I made experiments therefore (by favour) at Waterloo Bridge, extending a copper wire nine hundred and sixty feet in length upon the parapet of the bridge, and dropping from its extremities other wires with extensive plates of metal attached to them to complete contact with the water. The wire therefore and the water made one conducting circuit; and as the water ebbed or flowed with the tide, I hoped to obtain currents analogous to those of the brass ball (161.).

189. I constantly obtained deflections at the galvanometer, but they were

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* 1830. p. 399.

very irregular, and were in succession referred to other causes than that sought for. The different condition of the water as to purity on the two sides of the river; the difference in temperature; slight differences in the plates, in the solder used, in the more or less perfect contact made by twisting or otherwise; all produced effects in turn: and though I experimented on the water passing through the middle arches only; used platina plates instead of copper; and took every other precaution, I could not after three days obtain any satisfactory results.

190. Theoretically, it seems a necessary consequence that where water is flowing, there electric currents should be formed: thus, if a line be imagined passing from Dover to Calais through the sea, and returning through the land beneath the water to Dover, it traces out a circuit of conducting matter, one part of which, when the water moves up or down the channel, is cutting the magnetic curves of the earth, whilst the other is relatively at rest. This is a repetition of the wire experiment (171.), but with worse conductors. Still there is every reason to believe that electric currents do run in the general direction of the circuit described, either one way or the other, according as the passage of the waters is up or down the channel. Where the lateral extent of the moving water is enormously increased, it does not seem improbable that the effect should become sensible; and the gulf stream may thus, perhaps, from electric currents moving across it, by magneto-electric induction from the earth, exert a sensible influence upon the forms of the lines of magnetic variation*.

191. Though positive results have not yet been obtained by the action of the earth upon water and aqueous fluids, yet, as the experiments are very limited in their extent, and as such fluids do yield the current by artificial magnets (23.), (for transference of the current is proof that it may be produced (213.),) the supposition made, that the earth produces these induced currents within itself (181.) in consequence of its diurnal rotation, is still highly probable (222. 223.); and when it is considered that the moving masses extend for

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* Theoretically, even a ship or a boat when passing on the surface of the water, in northern or southern latitudes, should have currents of electricity running through it directly across the line of her motion; or if the water is flowing past the ship at anchor, similar currents should occur.

thousands of miles across the magnetic curves, cutting them in various directions within its mass, as well as at the surface, it is possible the electricity may rise to considerable intensity.

192. I hardly dare venture, even in the most hypothetical form, to ask whether the Aurora Borealis and Australis may not be the discharge of electricity, thus urged towards the poles of the earth, from whence it is endeavouring to return by natural and appointed means above the earth to the equatorial regions. The non-occurrence of it in very high latitudes is not at all against the supposition; and it is remarkable that Mr. Fox, who observed the deflections of the magnetic needle at Falmouth, by the Aurora Borealis, gives that direction of it which perfectly agrees with the present view. He states that all the variations at night were towards the east*, and this is what would happen if electric currents were setting from south to north in the earth under the needle, or from north to south in space above it.

§ 6. *General remarks and illustrations of the Force and Direction of Magneto-electric Induction.*

193. In the repetition and variation of Arago's experiment by Messrs. Babbage, Herschel, and Harris, those philosophers directed their attention to the differences of force observed amongst the metals and other substances in their action on the magnet. These differences were very great †, and led me to hope that by mechanical combinations of various metals important results might be obtained (183). The following experiments were therefore made, with a view to obtain, if possible, any such difference of the action of two metals.

194. A piece of soft iron bonnet-wire covered with cotton was laid bare and cleaned at one extremity, and there fastened by metallic contact with the clean end of a copper wire. Both wires were then twisted together like the strands of a rope, for eighteen or twenty inches; and the remaining parts being made to diverge, their extremities were connected with the wires of the galvanometer. The iron wire was about two feet long, the continuation to the galvanometer being copper.

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* Philosophical Transactions, 1831, p. 202.

† Ibid. 1825; p. 472, 1831, p. 78.

195. The twisted copper and iron (touching each other nowhere but at the extremity) was then passed between the poles of a powerful magnet arranged horse-shoe fashion (fig. 32.); but not the slightest effect was observed at the galvanometer, although the arrangement seemed fitted to show any electrical difference between the two metals relative to the action of the magnet.

196. A soft iron cylinder was then covered with paper at the middle part, and the twisted portion of the above compound wire coiled as a spiral around it, the connexion with the galvanometer still being made at the ends A and B. The iron cylinder was then brought in contact with the poles of a powerful magnet capable of raising thirty pounds; yet no signs of electricity appeared at the galvanometer. Every precaution was applied in making and breaking contact to accumulate effect, but no indications of a current could be obtained.

197. Copper and tin, copper and zinc, tin and zinc, tin and iron, and zinc and iron, were tried against each other in a similar manner (194), but not the slightest sign of electric currents could be procured.

198. Two flat spirals, one of copper and the other of iron, containing each eighteen inches of wire, were connected with each other and with the galvanometer, and then put face to face so as to be in contrary directions. When brought up to the magnetic pole (53.), no electrical indications at the galvanometer were observed. When one was turned round so that both were in the same direction, the effect at the galvanometer was very powerful.

199. The compound helix of copper and iron wire formerly described (8.) was arranged as a double helix, one of the helices being all iron and containing two hundred and fourteen feet, the other all copper and containing two hundred and eight feet. The two similar ends A A of the copper and iron helix were connected together, and the other ends B B of each helix connected with the galvanometer; so that when amagnet was introduced into the centre of the arrangement, the induced currents in the iron and copper would tend to proceed in contrary directions. Yet when a magnet was inserted, or a soft iron bar within made a magnet by contact with poles, no effect at the needle was produced.

200. A glass tube about fourteen inches long was filled with strong sulphuric acid. Twelve inches of the end of a clean copper wire were bent up

into a bundle and inserted into the tube, so as to make good superficial contact with the acid, and the rest of the wire passed along the outside of the tube and away to the galvanometer. A wire similarly bent up at the extremity was immersed in the other end of the sulphuric acid, and also connected with the galvanometer, so that the acid and copper wire were in the same parallel relation to each other in this experiment as iron and copper were in the first (194). When this arrangement was passed in a similar manner between the poles of the magnet, not the slightest effect at the galvanometer could be perceived.

201. From these experiments it would appear, that when metals of different kinds connected in one circuit are equally subject in every circumstance to magneto-electric induction, they exhibit exactly equal powers with respect to the currents which either are formed, or tend to form, in them. The same even appears to be the case with regard to fluids, and probably all other substances.

202. Still it seemed impossible that these results could indicate the relative inductive power of the magnet upon the different metals; for that the effect should be in some relation to the conducting power seemed a necessary consequence (139), and the influence of rotating plates upon magnets had been found to bear a general relation to the conducting power of the substance used.

203. In the experiments of rotation (81.), the electric current is excited and discharged in the same substance, be it a good or bad conductor; but in the experiments just described the current excited in iron could not be transmitted but through the copper, and that excited in copper had to pass through iron; i. e. supposing currents of dissimilar strength to be formed in the metals proportionate to their conducting power, the stronger current had to pass through the worst conductor, and the weaker current through the best.

204. Experiments were therefore made in which different metals insulated from each other were passed between the poles of the magnet, their opposite ends being connected with the same end of the galvanometer wire, so that the currents formed and led away to the galvanometer should oppose each other; and when considerable lengths of different wires were used, feeble deflections were obtained.

205. To obtain perfectly satisfactory results a new galvanometer was con-

structed, consisting of two independent coils, each containing eighteen feet of silked copper wire. These coils were exactly alike in shape and number of turns, and were fixed side by side with a small interval between them, in which a double needle could be hung by a fibre of silk exactly as in the former instrument (87.). The coils may be distinguished by the letters K L, and when electrical currents were sent through them in the same direction, acted upon the needle with the sum of their powers; when in opposite directions, with the difference of their powers.

206. The compound helix (199. 8.) was now connected, the ends A and B of the iron with A and B ends of galvanometer coil K, and the ends A and B of the copper with B and A ends of galvanometer coil L, so that the currents excited in the two helices should pass in opposite directions through the coils K and L. On introducing a small cylinder magnet within the helices, the galvanometer needle was powerfully deflected. On disuniting the iron helix, the magnet caused with the copper helix alone still stronger deflection in the same direction. On reuniting the iron helix, and unconnecting the copper helix, the magnet caused a moderate deflection in the contrary direction. Thus it was evident that the electric current induced by a magnet in a copper wire was far more powerful than the current induced by the same magnet in an equal iron wire.

207. To prevent any error that might arise from the greater influence, from vicinity or other circumstances, of one coil on the needle beyond that of the other, the iron and copper terminations were changed relative to the galvanometer coils K L, so that the one which before carried the current from the copper now conveyed that from the iron, and vice versâ. But the same striking superiority of the copper was manifested as before. This precaution was taken in the rest of the experiments with other metals to be described.

208. I then had wires of iron, zinc, copper, tin, and lead, drawn to the same diameter (very nearly one twentieth of an inch), and I compared exactly equal lengths, namely sixteen feet, of each in pairs in the following manner: The ends of the copper wire were connected with the ends A and B of galvanometer coil K, and the ends of the zinc wire with the terminations A and B of the galvanometer coil L. The middle part of each wire was then coiled six times round a cylinder of soft iron covered with paper, long enough to connect the

poles of Daniell's horse-shoe magnet (56.) (fig. 33.), so that similar helices of copper and zinc, each of six turns, surrounded the bar at two places equidistant from each other and from the poles of the magnet; but these helices were purposely arranged so as to be in contrary directions, and therefore send contrary currents through the galvanometer coils K and L.

209. On making and breaking contact between the soft iron bar and the poles of the magnet, the galvanometer was strongly affected; on detaching the zinc it was still more strongly affected in the same direction. On taking all the precautions before alluded to (207.), with others, it was abundantly proved that the current induced by the magnet in copper was far more powerful than in zinc.

210. The copper was then compared in a similar manner with tin, lead, and iron, and surpassed them all, even more than it did zinc. The zinc was then compared experimentally with the tin, lead, and iron, and found to produce a more powerful current than any of them. Iron in the same manner proved superior to tin and lead. Tin came next, and lead the last.

211. Thus the order of these metals is copper, zinc, iron, tin, and lead. It is exactly their order with respect to conducting power for electricity, and, with the exception of iron, is the order presented by the magneto-rotation experiments of Messrs. Babbage, Herschel, Harris, &c. The iron has additional power in the latter kind of experiments, because of its ordinary magnetic relations, and its place relative to magneto-electric action of the kind now under investigation cannot be ascertained by such trials. In the manner above described it may be correctly ascertained *.

212. It must still be observed that in these experiments the whole effect between different metals is not obtained; for of the thirty-four feet of wire included in each circuit, eighteen feet are copper in both, being the wire of the galvanometer coils; and as the whole circuit is concerned in the resulting force of the current, this circumstance must tend to diminish the difference which would appear between the metals if the circuits were of the same substances

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* Mr. Christie, who, being appointed reporter upon this paper, had it in his hands before it was complete, felt the difficulty (202.); and to satisfy his mind, made experiments upon iron and copper with the large magnet (44.), and came to the same conclusions as I have arrived at. The two set of experiments were perfectly independent of each other, neither of us being aware of the other's proceedings.

throughout. In the present case the difference obtained is probably not more than a half of that which would be given if the whole of each circuit were of one metal.

213. These results tend to prove that the currents produced by magneto-electric induction in bodies is proportional to their conducting power. That they are *exactly* proportional to and altogether dependent upon the conducting power, is, I think, proved by the perfect neutrality displayed when two metals or other substances, as acid, water, &c. &c. (201. 186.), are opposed to each other in their action. The feeble current which tends to be produced in the worse conductor, has its transmission favoured in the better conductor, and the stronger current which tends to form in the latter has its intensity diminished by the obstruction of the former; and the forces of generation and obstruction are so perfectly balanced as to neutralize each other exactly. Now as the obstruction is inversely as the conducting power, the tendency to generate a current must be directly as that power to produce this perfect equilibrium.

214. The cause of the equality of action under the various circumstances described, where great extent of wire (183.) or wire and water (184.) were connected together, which yet produced such different effects upon the magnet, is now evident and simple.

215. The effects of a rotating substance upon a needle or magnet ought, where ordinary magnetism has no influence, to be directly as the conducting power of the substance; and I venture now to predict that such will be found to be the case; and that in all those instances where non-conductors have been supposed to exhibit this peculiar influence, the motion has been due to some interfering cause of an ordinary kind; as mechanical communication of motion through the parts of the apparatus, or otherwise (as in the case Mr. Harris has pointed out[*]); or else to ordinary magnetic attractions. To distinguish the effects of the latter from those of the induced electric currents, I have been able to devise a most perfect test, which shall be almost immediately described (243.).

216. There is every reason to believe that the magnet or magnetic needle will become an excellent measurer of the conducting power of substances

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[*] Philosophical Transactions, 1831. p. 68.

rotated near it; for I have found by careful experiment, that when a constant current of electricity was sent successively through a series of wires of copper, platina, zinc, silver, lead, and tin, drawn to the same diameter; the deflection of the needle was exactly equal by them all. It must be remembered that when bodies are rotated in a horizontal plane, the magnetism of the earth is active upon them. As the effect is general to the whole of the plate, it may not interfere in these cases; but in some experiments and calculations may be of important consequence.

217. Another point which I endeavoured to ascertain, was, whether it was essential or not that the moving part of the wire should, in cutting the magnetic curves, pass into positions of greater or lesser magnetic force; or whether, always intersecting curves of equal magnetic intensity, the mere motion was sufficient for the production of the current. That the latter is true, has been proved already in several of the experiments on terrestrial magneto-electric induction. Thus the electricity evolved from the copper plate (149.), the currents produced in the rotating globe (161, &c.), and those passing through the moving wire (171.), are all produced under circumstances in which the magnetic force could not but be the same during the whole experiment.

218. To prove the point with an ordinary magnet, a copper disc was cemented upon the end of a cylinder magnet, with paper intervening; the magnet and disc were rotated together, and collectors (attached to the galvanometer) brought in contact with the circumference and the central part of the copper plate. The galvanometer needle moved as in former cases, and the *direction* of motion was the *same* as that which would have resulted, if the copper only had revolved, and the magnet been fixed. Neither was there any apparent difference in the quantity of deflection. Hence, rotating the magnet causes no difference in the results; for a rotatory and a stationary magnet produce the same effect upon the moving copper.

219. A copper cylinder, closed at one extremity, was then put over the magnet, one half of which it inclosed like a cap; it was firmly fixed, and prevented from touching the magnet anywhere by interposed paper. The arrangement was then floated in a narrow jar of mercury, so that the lower edge of the copper cylinder touched the fluid metal; one wire of the galvanometer dipped into this mercury, and the other into a little cavity in the centre of the

end of the copper cap. Upon rotating the magnet and its attached cylinder, abundance of electricity passed through the galvanometer, and in the same direction as if the cylinder had rotated only, the magnet being still. The results therefore were the same as those with the disc (218.).

220. That the metal of the magnet itself might be substituted for the moving cylinder, disc, or wire, seemed an inevitable consequence, and yet one which would exhibit the effects of magneto-electric induction in a striking form. A cylinder magnet had therefore a little hole made in the centre of each end to receive a drop of mercury, and was then floated pole upwards in the same metal contained in a narrow jar. One wire from the galvanometer dipped into the mercury of the jar, and the other into the drop contained in the hole at the upper extremity of the axis. The magnet was then revolved by a piece of string passed round it, and the galvanometer-needle immediately indicated a powerful current of electricity. On reversing the order of rotation, the electrical current was reversed. The direction of the electricity was the same as if the copper cylinder (219.) or a copper wire had revolved round the fixed magnet in the same direction as that which the magnet itself had followed. Thus a singular independence of the magnetism and the bar in which it resides is rendered evident.

221. In the above experiment the mercury reached about half way up the magnet; but when its quantity was increased until within one eighth of an inch of the top, or diminished until equally near the bottom, still the same effects and the *same direction* of electrical current was obtained. But in those extreme proportions the effects did not appear so strong as when the surface of the mercury was about the middle, or between that and an inch from each end. The magnet was eight inches and a half long, and three quarters of an inch in diameter.

222. Upon inversion of the magnet, and causing rotation in the same direction, i. e. always screw or always unscrew, then a contrary current of electricity was produced. But when the motion of the magnet was continued in a direction constant in relation to its *own axis*, then electricity of the same kind was collected at both poles, and the opposite electricity at the equator, or in its neighbourhood, or in the parts corresponding to it. If the magnet be held parallel to the axis of the earth, with its unmarked pole directed to the

pole star, and then rotated so that its upper parts pass from west to east in conformity to the motion of the earth; then positive electricity may be collected at the extremities of the magnet, and negative electricity at or about the middle of its mass.

223. When the galvanometer was very sensible, the mere spinning of the magnet in the air, whilst one of the galvanometer wires touched the extremity, and the other the equatorial parts, was sufficient to evolve a current of electricity and deflect the needle.

224. Experiments were then made with a similar magnet, for the purpose of ascertaining whether any return of the electric current could occur at the central or axial parts, they having the same angular velocity of rotation as the other parts (259.); the belief being that it could not.

225. A cylinder magnet, seven inches in length, and three quarters of an inch in diameter, had a hole pierced in the direction of its axis from one extremity, a quarter of an inch in diameter, and three inches deep. A copper cylinder, surrounded by paper and amalgamated at both extremities, was fixed in the hole so as to be in metallic contact at the bottom, by a little mercury, with the middle of the magnet; insulated at the sides by the paper; and projecting about a quarter of an inch above the end of the steel. A quill was put over the copper rod, which reached to the paper, and formed a cup to receive mercury for the completion of the contact. A high paper edge was also raised round that end of the magnet, and mercury put within it, which however had no metallic connexion with that in the quill, except through the magnet itself and the copper rod (fig. 34.). The wires A and B from the galvanometer were dipped into these two portions of mercury; any current through them could, therefore, only pass down the magnet towards its equatorial parts, and then up the copper rod; or vice versâ.

226. When thus arranged and rotated screw fashion, the marked end of the galvanometer needle went west, indicating that there was a current through the instrument from A to B, and consequently from B through the magnet and copper rod to A (fig. 34.).

227. The magnet was then put into a jar of mercury (fig. 35.) as before (219.); the wire A left in contact with the copper axis, but the wire B dipped in the mercury of the jar, and therefore in metallic communication with the

equatorial parts of the magnet instead of its polar extremity. On revolving the magnet screw fashion, the galvanometer needle was deflected in the same direction as before, but far more powerfully. Yet it is evident that the parts of the magnet from the equator to the pole were out of the electric circuit.

228. Then the wire A was connected with the mercury on the extremity of the magnet, the wire B still remaining in contact with that in the jar (fig. 36.), so that the copper axis was altogether out of the circuit. The magnet was again revolved screw fashion, and again caused the same deflection of the needle, the current being as strong as it was in the last trial (227.), and much stronger than at first (226.).

229. Hence it is evident that there is no discharge of the current at the centre of the magnet, for the current, now freely evolved, is up through the magnet; but in the first experiment (226.), it was down. In fact, at that time, it was only the part of the moving metal equal to a little disc extending from the end of the wire B in the mercury to the wire A that was efficient, i. e. moving with a different angular velocity to the rest of the circuit (258.); and for that portion the direction of the current is consistent with the other results.

230. In the two after experiments, the *lateral* parts of the magnet or of the copper rod are those which move relative to the other parts of the circuit, i. e. the galvanometer wires; and being more extensive, intersecting more curves; or moving with more velocity, produce the greater effect. For the discal part, the direction of the induced electric current is the same in all, namely, from the circumference towards the centre.

231. The law under which the induced electric current excited in bodies moving relatively to magnets, is made dependent on the intersection of the magnetic curves by the metal (114.) being thus rendered more precise and definite (217. 220. 224.), seemed now even to apply to the cause in the first section of the former paper; and by rendering a perfect reason for the effects produced, take away any for supposing that peculiar condition, which I ventured to call the electro-tonic state (60.).

232. When an electrical current is passed through a wire, that wire is surrounded at every part by magnetic curves, diminishing in intensity according to their distance from the wire, and which in idea may be likened to rings situated in planes perpendicular to the wire or rather to the electric current

within it. These curves, although different in form, are perfectly analogous to those existing between two contrary magnetic poles opposed to each other; and when a second wire, parallel to that which carries the current, is made to approach the latter (18.), it passes through magnetic curves exactly of the same kind as those it would intersect when carried between opposite magnetic poles (109.), in one direction; and as it recedes from the inducing wire, it cuts the curves around it in the same manner that it would do those between the same poles if moved in the other direction.

233. If the wire N P (fig. 40.) have an electric current passed through it in the direction from P to N, then the dotted ring may represent a magnetic curve round it, and it is in such a direction that if small magnetic needles be placed as tangents to it, they will become arranged as in the figure, *n* and *s* indicating north and south ends (44. *note.*).

234. But if the current of electricity were made to cease for a while, and magnetic poles were used instead to give direction to the needles, and make them take the same position as when under the influence of the current, then they must be arranged as at fig. 41; the marked and unmarked poles *a b* above the wire, being in opposite directions to those *a′ b′* below. In such a position therefore the magnetic curves between the poles *a b* and *a′ b′* have the same general direction with the corresponding parts of the ring magnetic curve surrounding the wire N P carrying an electric current.

235. If the second wire *p n* (fig. 40.), be now brought towards the principal wire, carrying a current, it will cut an infinity of magnetic curves, similar in direction to that figured, and consequently similar in direction to those between the poles *a b* of the magnets (fig. 41.), and it will intersect these current curves in the same manner as it would the magnet curves, if it passed from above between the poles downwards. Now, such an intersection would, with the magnets, induce an electric current in the wire from *p* to *n* (114.); and therefore as the curves are alike in arrangement, the same effect ought to result from the intersection of the magnetic curves dependent on the current in the wire N P; and such is the case, for on approximation the induced current is in the opposite direction to the principal current (19.).

236. If the wire *p′ n′* be carried up from below, it will pass in the opposite direction between the magnetic poles; but then also the magnetic poles them-

selves are reversed (fig. 41.), and the induced current is therefore (114.) still in the same direction as before. It is also, for equally sufficient and evident reasons, in the same direction, if produced by the influence of the curves dependent upon the wire.

237. When the second wire is retained at rest in the vicinity of the principal wire, no current is induced through it, for it is intersecting no magnetic curves. When it is removed from the principal wire, it intersects the curves in the opposite direction to what it did before (235.); and a current in the opposite direction is induced, which therefore corresponds with the direction of the principal current (19.). The same effect would take place if by inverting the direction of motion of the wire in passing between either set of poles (fig. 41.), it were made to intersect the curves there existing in the opposite direction to what it did before.

238. In the first experiments (10. 13.), the inducing wire and that under induction were arranged at a fixed distance from each other, and then an electric current sent through the former. In such cases the magnetic curves themselves must be considered as moving (if I may use the expression) across the wire under induction, from the moment at which they begin to be developed until the magnetic force of the current is at its utmost; expanding as it were from the wire outwards, and consequently being in the same relation to the fixed wire under induction as if *it* had moved in the opposite direction across them, or towards the wire carrying the current. Hence the first current induced in such cases was in the contrary direction to the principal current (17. 235.). On breaking the battery contact, the magnetic curves (which are mere expressions for arranged magnetic forces) may be conceived as contracting upon and returning towards the failing electrical current, and therefore move in the opposite direction across the wire, and cause an opposite induced current to the first.

239. When, in experiments with ordinary magnets, the latter, in place of being moved past the wires, were actually made near them (27. 36.), then a similar progressive development of the magnetic curves may be considered as having taken place, producing the effects which would have occurred by motion of the wires in one direction; the destruction of the magnetic power corresponds to the motion of the wire in the opposite direction.

240. If, instead of intersecting the magnetic curves of a straight wire carrying a current, by approximating or removing a second wire (235.), a revolving plate be used, being placed for that purpose near the wire, and, as it were, amongst the magnetic curves, then it ought to have continuous electric currents induced within it; and if a line joining the wire with the centre of the plate were perpendicular to both, then the induced current ought to be, according to the law (114.), directly across the plate, from one side to the other, and at right angles to the direction of the inducing current.

241. A single metallic wire one twentieth of an inch in diameter had an electric current passed through it, and a small copper disc one inch and a half in diameter revolved near to and under, but not in actual contact with it (fig. 39.). Collectors were then applied at the opposite edges of the disc, and wires from them connected with the galvanometer. As the disc revolved in one direction, the needle was deflected on one side; and when the direction of revolution was reversed, the needle was inclined on the other side, in accordance with the results anticipated.

242. Thus the reasons which induced me to suppose a particular state in the wire (60.) have disappeared; and though it still seems to me unlikely that a wire at rest in the neighbourhood of another carrying a powerful electric current is entirely indifferent to it, yet I am not aware of any distinct *facts* which authorize the conclusion that it is in a particular state.

243. In considering the nature of the cause assigned in these papers to account for the mutual influence of magnets and moving metals (120.), and comparing it with that heretofore admitted, namely, the induction of a feeble magnetism like that produced in iron, it occurred to me that a most decisive experimental test of the two views could be applied (215.).

244. No other known power has like direction with that exerted between an electric current and a magnetic pole; it is tangential, while all other forces, acting at a distance, are direct. Hence, if a magnetic pole on one side of a revolving plate follow its course by reason of its obedience to the tangential force exerted upon it by the very current of electricity which it has itself caused, a similar pole on the opposite side of the plate should immediately set it free from this force; for the currents which tend to be formed by the action of the two poles are in opposite directions; or rather no current tends to be formed,

or no magnetic curves are intersected (114.); and therefore the magnet should remain at rest. On the contrary, if the action of a north magnetic pole were to produce a southness in the nearest part of the copper plate, and a diffuse northness elsewhere (82.), as is really the case with iron; then the use of another north pole on the opposite side of the same part of the plate should double the effect instead of destroying it, and double the tendency of the first magnet to move with the plate.

245. A thick copper plate (85.) was therefore fixed on a vertical axis, a bar magnet was suspended by a platted silk cord, so that its marked pole hung over the edge of the plate, and a sheet of paper being interposed, the plate was revolved; immediately the magnetic pole obeyed its motion and passed off in the same direction. A second magnet of equal size and strength was then suspended to the first, so that its marked pole should hang *beneath* the edge of the copper plate in a corresponding position to that above, and at an equal distance (fig. 37.). Then a paper sheath or screen being interposed as before, and the plate revolved, the poles were found entirely indifferent to its motion, although either of them alone would have followed the course of rotation.

246. On turning one magnet round, so that *opposite* poles were on each side of the plate, then the mutual action of the poles and the moving metal was a maximum.

247. On suspending one magnet so that its axis was level with the plate, and either pole opposite its edge, the revolution of the plate caused no motion of the magnet. The electrical currents dependent upon induction would now tend to be produced in a vertical direction across the thickness of the plate, but could not be so discharged, at least only to so slight a degree as to leave all effects insensible; but ordinary magnetic induction, or that on an iron plate, would be equally if not more powerfully developed in such a position (251.).

248. Then, with regard to the production of electricity in these cases:—whenever motion was communicated by the plate to the magnets, currents existed; when it was not communicated, they ceased. A marked pole of a large bar magnet was put under the edge of the plate; collectors (86.) applied at the axis and edge of the plate as on former occasions (fig. 38.), and these connected with the galvanometer; when the plate was revolved, abundance of electricity passed to the instrument. The unmarked pole of a similar magnet was then

put over the place of the former pole, so that contrary poles were above and below; on revolving the plate, the electricity was more powerful than before. The latter magnet was then turned end for end, so that marked poles were both above and below the plate, and then, upon revolving it, scarcely any electricity was procured. By adjusting the distance of the poles so as to correspond with their relative force, they at last were brought so perfectly to neutralize each other's inductive action upon the plate, that no electricity could be obtained with the most rapid motion.

249. I now proceeded to compare the effect of similar and dissimilar poles upon iron and copper, adopting for the purpose Mr. STURGEON's very useful form of ARAGO's experiment. This consists in a circular plate of metal supported in a vertical plane by a horizontal axis, and weighted a little at one edge or rendered excentric so as to vibrate like a pendulum. The poles of the magnets are applied near the side and edges of these plates, and then the number of vibrations, required to reduce the vibrating arc a certain constant quantity, noted. In the first description of this instrument * it is said that opposite poles produced the greatest retarding effect, and similar poles none; and yet within a page of the place the effect is considered as of the same kind with that produced in iron.

250. I had two such plates mounted, one of copper, one of iron. The copper plate alone gave sixty vibrations, in the average of several experiments, before the arc of vibration was reduced from one constant mark to another. On putting opposite magnetic poles near to, and on each side of, the same place, the vibrations were reduced to fifteen. On putting similar poles on each side of it, they rose to fifty; and on putting two pieces of wood of equal size with the poles equally near, they became fifty-two. So that, when similar poles were used, the magnetic effect was little or none, (the obstruction being due to the confinement of the air, rather,) whilst with opposite poles it was the greatest possible. When a pole was presented to the edge of the plate, no retardation occurred.

251. The iron plate alone made thirty-two vibrations, whilst the arc of vibration diminished a certain quantity. On presenting a magnetic pole to the edge of the plate (247.), the vibrations were diminished to eleven; and when the pole was about half an inch from the edge, to five.

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* Edin. Phil. Journal, 1825. p. 124.

252. When the marked pole was put at the side of the iron plate at a certain distance, the number of vibrations was only five. When the marked pole of the second bar was put on the opposite side of the plate at the same distance (250.), the vibrations were reduced to two. But when the second pole was an unmarked one, yet occupying exactly the same position, the vibrations rose to twenty-two. By removing the stronger of these two opposite poles a little way from the plate, the vibrations increased to thirty-one, or nearly the original number. But on removing it *altogether*, they fell to between five and six.

253. Nothing can be more clear, therefore, than that with iron, and bodies admitting of ordinary magnetic induction, *opposite* poles on opposite sides of the edge of the plate neutralize each other's effect, whilst *similar* poles exalt the action; a single pole end on is also sufficient. But with copper, and substances not sensible to ordinary magnetic impressions, *similar* poles on opposite sides of the plate neutralize each other; *opposite* poles exalt the action; and a single pole at the edge or end on does nothing.

254. Nothing can more completely show the thorough independence of the effects obtained with the metals by ARAGO, and those due to ordinary magnetic forces; and henceforth, therefore, the application of two poles to various moving substances will, if they appear at all magnetically affected, afford a proof of the nature of that affection. If opposite poles produce more effect than one, the force will be due to electric currents. If similar poles produce more effect than one, then the power is *not* electrical: it will not be like that active in the metals and carbon when moving, and in most cases will probably be found to be not even magnetical, but the result of irregular causes not anticipated and guarded against.

255. The result of these investigations tends to show that there are really but very few bodies that are magnetic in the manner of iron. I have often sought for indications of this power in the common metals and other substances; and once in illustration of ARAGO's objection (82.), and in hopes of ascertaining the existence of currents in metals by the momentary approach of a magnet, suspended a disc of copper by a single fibre of silk in an excellent vacuum, and approximated powerful magnets on the outside of the jar, making them approach and recede in unison with a pendulum that vibrated as the disc would do: but no motion could be obtained; not merely, no indication of

ordinary magnetic powers, but none of *any electric current* occasioned in the metal by the approximation and recession of the magnet. I therefore venture to arrange substances in three classes as regards their relation to magnets; first, those which are affected when at rest, like iron, nickel, &c. being such as possess ordinary magnetic properties; then, those which are affected when in motion, being conductors of electricity in which are produced electric currents by the inductive force of the magnet; and, lastly, those which are perfectly indifferent to the magnet, whether at rest or in motion.

256. Although it will require further research, and probably close investigation, both experimental and mathematical, before the exact mode of action between a magnet and metal moving relatively to each other is ascertained; yet many of the results appear sufficiently clear and simple to allow of expression in a somewhat general manner. If a terminated wire move so as to cut a magnetic curve, a power is called into action which tends to urge an electric current through it; but this current cannot be brought into existence unless provision be made at the ends of the wire for its discharge and renewal.

257. If a second wire move in the same direction as the first, the same power is exerted upon it, and it is therefore unable to alter the condition of the first: for there appear to be no natural differences among substances when connected in a series, by which, when moving under the same circumstances relative to the magnet, one tends to produce a more powerful electric current in the whole circuit than another (201. 214.).

258. But if the second wire move with a different velocity, or in some other direction, then variations in the force exerted take place; and if connected at their extremities, an electric current passes through them.

259. Taking, then, a mass of metal or an endless wire, and referring to the pole of the magnet as a centre of action, (which though perhaps not strictly correct may be allowed for facility of expression, at present,) if all parts move in the same direction, and with the same angular velocity, and through magnetic curves of constant intensity, then no electric currents are produced. This point is easily observed with masses subject to the earth's magnetism, and may be proved with regard to small magnets; by rotating them, and leaving the metallic arrangements stationary, no current is produced.

260. If one part of the wire or metal cut the magnetic curves, whilst the other

is stationary, then currents are produced. All the results obtained with the galvanometer are more or less of this nature, the galvanometer extremity being the fixed part. Even those with the wire, galvanometer, and earth (170.), may be considered so without any error in the result.

261. If the motion of the metal be in the same direction, but the angular velocity of its parts relative to the pole of the magnet different, then currents exist. This is the case in ARAGO's experiment, and also in the wire subject to the earth's induction (172.), when it was moved from west to east.

262. If the magnet moves not directly to or from the arrangement, but laterally, then the case is similar to the last.

263. If different parts move in opposite directions across the magnetic curves, then the effect is a maximum for equal velocities.

264. All these in fact are variations of one simple condition, namely, that all parts of the mass shall not move in the same direction across the curves, and with the same angular velocity. But they are forms of expression which being retained in the mind, I have found useful when comparing the consistency of particular phenomena with general results.

Royal Institution,
December 21, 1831.

VII.—*On the Theory of the Perturbations of the Planets.* By JAMES IVORY, *A.M. F.R.S. Instit. Reg. Sc. Paris. Corresp. et Reg. Sc. Gottin. Corresp.*

Read January 19, 1832.

THE perturbations of the planets is the subject of reiterated researches by all the great geometers who have raised up Physical Astronomy to its present elevation. They have been successful in determining the variations which the elements of the orbit of a disturbed planet undergo; and in expressing these variations analytically, in the manner best adapted for computation. But the inquirer who turns his attention to this branch of study will find that it is made to depend upon a theory in mechanics, which is one of considerable analytical intricacy, known by the name of the Variation of the Arbitrary Constants. Considerations similar to those employed in this theory were found necessary in Physical Astronomy from its origin; but the genius of LAGRANGE imagined and completed the analytical processes of general application. In a dynamical problem which is capable of an exact solution, such as a planet revolving by the central attraction of the sun, the formulas constructed by LAGRANGE enable us to ascertain the alterations that will be induced on the original motions of the body, if we suppose it urged by new and very small forces, such as the irregular attractions of the other bodies of the planetary system. General views of this nature are very valuable, and contribute greatly to the advancement of science. But their application is sometimes attended with inconvenience. In particular cases, the general structure of the formulas may require a long train of calculation, in order to extricate the values of the quantities sought. It may be necessary for attaining this end to pass through many differential equations, and to submit to much subordinate calculation. The remedy for this inconvenience seems to lie in separating the general principles from the analytical processes by which they are carried into effect. In some important problems, a great advantage, both in brevity and clearness, will be obtained by adapting the investigation to the particular circumstance of the case, and attending solely to the principles of the method in deducing the solution. It may therefore become a question whether it be not possible to simplify physical astronomy by calling in the aid only of the usual principles of dynamics, and by setting aside every formula or equation not absolutely necessary for arriving at the final results. The utility of such an attempt, if successful, can hardly be doubted. By rendering more accessible a subject of great interest and importance, the study of English mathematicians may be recalled to a theory which, although it originated in England, has not received the attention it deserves, and which it *has* met with in foreign countries.

The paper which I have the honour to submit to the Royal Society, contains a complete determination of the variable elements of the elliptic orbit of a disturbed planet, deduced from three differential equations that follow readily from the mechanical conditions of the problem. In applying these equations, the procedure is the same whether a planet is urged by the sole action of the central force of the sun, or is besides disturbed by the attraction of other bodies revolving about that luminary; the only difference being that, in the first case, the elements of the orbit are all constant, whereas in the other case they are all variable. The success of the method here followed is derived from a new differential equation between the time and the area described by the planet in its momentary plane, which greatly shortens the investigation by making it unnecessary to consider the projection of the orbit. But the solution in this paper, although no reference is made to the analytical formulas of the theory of the variation of the arbitrary constants, is no less an application of that method, and an example of its utility and of the necessity of employing it in very complicated problems.

1. If S represent the sun and P, P′ two planets circulating round that luminary, it is proposed to investigate the effect of the attraction of P′ to disturb the motion of P and to change the elements of its orbit. We here confine our attention to one disturbing planet; for there is no difficulty in extending to any number, the conclusions that shall be established in the case of one.

The positions of the planets P and P′ may be ascertained as usual by the rectangular coordinates x, y, z and x', y', z'; x, y, x', y' being contained in a plane passing through the origin of the coordinates placed in the sun's centre; and z, z' being perpendicular to the same plane.

Further, let M, m, m' denote the respective masses of S, P, P′; r and r' the distance of P and P′ from S, and ϱ the distance between the two planets; then, putting $\mu = \text{M} + m$, the direct attraction between S and P will be $\frac{\mu}{r^2}$; and the resolved parts of this force, acting in the respective directions of x, y, z, and tending to diminish these lines, will be

$$\frac{\mu x}{r^3},\ \frac{\mu y}{r^3},\ \frac{\mu z}{r^3}.$$

The planet P′ attracts S with a force $= \frac{m'}{r'^2}$, of which the resolved parts are,

$$\frac{m' x'}{r'^3},\ \frac{m' y'}{r'^3},\ \frac{m' z'}{r'^3}.$$

The same planet P′ attracts P with the force $\frac{m'}{\varrho^2}$, of which the partial forces are

$$\frac{m'(x'-x)}{\varrho^3},\ \frac{m'(y'-y)}{\varrho^3},\ \frac{m'(z'-z)}{\varrho^3}.$$

Were S and P attracted by P′ in like directions with equal intensity, the relative situation of the two bodies would not be changed, and the action of P′ might be neglected: but the attractions parallel to the coordinates being unequal, the differences of these attractions, viz.

$$\frac{m'(x'-x)}{\varrho^3} - \frac{m' x'}{r'^3},\ \frac{m'(y'-y)}{\varrho^3} - \frac{m' y'}{r'^3},\ \frac{m'(z'-z)}{\varrho^3} - \frac{m' z'}{r'^3},$$

are exerted in altering the place of P relatively to S. These last forces increase the coordinates x, y, z; and, therefore, they must be subtracted from the former forces which have opposite directions, in order to obtain the total forces acting in the directions of the coordinates and affecting the motion of P relatively to S, viz.

$$\frac{m x}{r^3} - \frac{m'(x'-x)}{\varrho^3} + \frac{m' x'}{r'^3},$$

$$\frac{m y}{r^3} - \frac{m'(y'-y)}{\varrho^3} + \frac{m' y'}{r'^3},$$

$$\frac{m z}{r^3} - \frac{m'(z'-z)}{\varrho^3} + \frac{m' z'}{r'^3}.$$

But, if dt represent the element of the time supposed to flow uniformly, the actual velocities with which the coordinates increase are, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$; and the increments of these velocities, $\frac{ddx}{dt^2}, \frac{ddy}{dt^2}, \frac{ddz}{dt^2}$, are the effects produced by all the forces that urge the planet. Equating now the forces really in action to the measure of the effects they produce, and observing that the two equivalent quantities have been estimated in opposite directions, we obtain the following equations for determining the place of P relatively to S at any proposed instant of time,

$$\frac{ddx}{dt^2} + \frac{\mu x}{r^3} = \frac{m'(x'-x)}{\rho^3} - \frac{m'x'}{r'^3},$$

$$\frac{ddy}{dt^2} + \frac{\mu y}{r^3} = \frac{m'(y'-y)}{\rho^3} - \frac{m'y'}{r'^3},$$

$$\frac{ddz}{dt^2} + \frac{\mu z}{r^3} = \frac{m'(z'-z)}{\rho^3} - \frac{m'z'}{r'^3}.$$

If we now assume

$$\mathrm{R} = \frac{m'}{\mu} \times \left\{ \frac{1}{\sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}} - \frac{xx' + yy' + zz'}{(x'^2 + y'^2 + z'^2)^{\frac{3}{2}}} \right\},$$

it will be found that the partial differentials, $\mu \times \frac{d\mathrm{R}}{dx}, \mu \times \frac{d\mathrm{R}}{dy}, \mu \times \frac{d\mathrm{R}}{dz}$, are respectively equal to the quantities on the right sides of the last equations, that is, to the disturbing forces tending to increase the coordinates x, y, z. These equations may therefore be thus written,

$$\left.\begin{aligned} \frac{ddx}{\mu dt^2} + \frac{x}{r^3} &= \frac{d\mathrm{R}}{dx}, \\ \frac{ddy}{\mu dt^2} + \frac{y}{r^3} &= \frac{d\mathrm{R}}{dy}, \\ \frac{ddz}{\mu dt^2} + \frac{z}{r^3} &= \frac{d\mathrm{R}}{dz}, \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (\mathrm{A})$$

If it be asked, What notion must be affixed to the symbol μdt^2?, it will be recollected that $\frac{\mu}{r^2}$ is the attraction between S and P at the distance r; and if we suppose that P describes a circle, of which unit is the radius, round S, the centripetal force in the circle will be $\frac{\mu}{1^2}$ or μ; and the velocity with which P moves in the circle will be proportional to $\sqrt{\mu}$. Thus the algebraic quantities $t\sqrt{\mu}$ and $dt\sqrt{\mu}$ represent the arcs of this circular orbit, which are described in the times t and dt.

It is requisite in what follows to transform the coordinates x, y, z into other variable quantities better adapted for use in astronomy. Let λ and λ' denote the longitudes of the planets P and P′ reckoned in the fixt plane of $x\,y$, and s and s' the tangents of their latitudes, that is, of the angles which the radii vectores r and r' make with the same plane: then,

$$x = \frac{r\cos\lambda}{\sqrt{1+s^2}}, \quad x' = \frac{r'\cos\lambda'}{\sqrt{1+s'^2}},$$

$$y = \frac{r\sin\lambda}{\sqrt{1+s^2}}, \quad y' = \frac{r'\sin\lambda'}{\sqrt{1+s'^2}},$$

$$z = \frac{r\,s}{\sqrt{1+s^2}}, \quad z' = \frac{r'\,s'}{\sqrt{1+s'^2}}.$$

In the transformations alluded to, the quantities $\frac{d\,\mathrm{R}}{dx}, \frac{d\,\mathrm{R}}{dy}, \frac{d\,\mathrm{R}}{dz}$ must be expressed in the partial differentials of R relatively to the new variables r, λ, s; and it will conduce to clearness of method if these calculations be dispatched here. We have the equation,

$$\frac{d\,\mathrm{R}}{dx} = \frac{d\,\mathrm{R}}{dr}\cdot\frac{dr}{dx} + \frac{d\,\mathrm{R}}{d\lambda}\cdot\frac{d\lambda}{dx} + \frac{d\,\mathrm{R}}{ds}\cdot\frac{ds}{dx};$$

and having computed the differentials $\frac{dr}{dx}, \frac{d\lambda}{dx}, \frac{ds}{dx}$ from the formulas

$$r = \sqrt{x^2+y^2+z^2}, \qquad \tan.\lambda = \frac{y}{x}, \qquad s = \frac{z}{\sqrt{x^2+y^2}},$$

the substitution of the results will make known the expression of $\frac{d\,\mathrm{R}}{dx}$. By the like procedure the values of $\frac{d\,\mathrm{R}}{dy}$ and $\frac{d\,\mathrm{R}}{dz}$ will be found

$$\left.\begin{aligned}
\frac{d\,\mathrm{R}}{dx} &= \frac{d\,\mathrm{R}}{dr}\cdot\frac{\cos\lambda}{\sqrt{1+s^2}} - \frac{d\,\mathrm{R}}{d\lambda}\cdot\frac{\sin\lambda\sqrt{1+s^2}}{r} - \frac{d\,\mathrm{R}}{ds}\cdot\frac{\cos\lambda\, s\sqrt{1+s^2}}{r},\\
\frac{d\,\mathrm{R}}{dy} &= \frac{d\,\mathrm{R}}{dr}\cdot\frac{\sin\lambda}{\sqrt{1+s^2}} + \frac{d\,\mathrm{R}}{d\lambda}\cdot\frac{\cos\lambda\sqrt{1+s^2}}{r} - \frac{d\,\mathrm{R}}{ds}\cdot\frac{\sin\lambda\, s\sqrt{1+s^2}}{r},\\
\frac{d\,\mathrm{R}}{dz} &= \frac{d\,\mathrm{R}}{dr}\cdot\frac{s}{\sqrt{1+s^2}} \qquad\qquad\qquad\qquad\quad + \frac{d\,\mathrm{R}}{ds}\cdot\frac{\sqrt{1+s^2}}{r}.
\end{aligned}\right\}\;.\;.\;.\;(\mathrm{B})$$

The new partial differentials of R represent the disturbing forces reduced to new directions. By combining the formulas (B), we get

$$\frac{d\mathrm{R}}{dr} = \frac{d\mathrm{R}}{dx} \cdot \frac{\cos\lambda}{\sqrt{1+s^2}} + \frac{d\mathrm{R}}{dy} \cdot \frac{\sin\lambda}{\sqrt{1+s^2}} + \frac{d\mathrm{R}}{dz} \cdot \frac{s}{\sqrt{1+s^2}};$$

and it will readily appear that the coefficients of $\frac{d\mathrm{R}}{dx}$, $\frac{d\mathrm{R}}{dy}$, $\frac{d\mathrm{R}}{dz}$ are the respective cosines of the angles which the directions of the forces make with r; so that $\frac{d\mathrm{R}}{dr}$ is the sum of the three partial forces that urge the planet from the sun. In like manner it may be proved that $\frac{d\mathrm{R}}{d\lambda} \cdot \frac{\sqrt{1+s^2}}{r}$ is the disturbing force perpendicular to the plane passing through the sun and the coordinate z, that is, to the circle of latitude; and that $\frac{d\mathrm{R}}{ds} \cdot \frac{1+s^2}{r}$ is the force acting in the same plane perpendicular to r, and tending to increase the latitude.

2. If the equations (A), after being multiplied by $2\,dx$, $2\,dy$, $2\,dz$, be added together, and then integrated, we shall get this well-known result,

$$\frac{dx^2+dy^2+dz^2}{\mu \,.\, dt^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'\mathrm{R}, \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1)$$

in which $\frac{1}{a}$ is the arbitrary constant, and the symbol $d'\mathrm{R}$ is put for

$$\frac{d\mathrm{R}}{dx}\,dx + \frac{d\mathrm{R}}{dy}\,dy + \frac{d\mathrm{R}}{dz}\,dz;$$

that is, for the differential of R, on the supposition that x, y, z, the coordinates of the disturbed planet, are alone variable. If we conceive that R is transformed into a function of the other quantities r, λ, s, we shall therefore have

$$d'\mathrm{R} = \frac{d\mathrm{R}}{dr}\,dr + \frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds.$$

Supposing that the radius vector r, at the end of the small interval of time dt, becomes equal to $r + dr$, and that dv expresses the small angle contained between r and $r + dr$, we shall have

$$dr^2 + r^2 . dv^2 = dx^2 + dy^2 + dz^2;$$

for each of these quantities is equal to the square of the small portion of its orbit which the planet describes in the time dt. The last equation may therefore be thus written,

$$\frac{dr^2}{\mu\, dt^2} + \frac{r^2 dv^2}{\mu\, dt^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'\mathrm{R}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

The double of the small area contained between the radii r and $r + dr$, is equal to $r^2 dv$; and as x, y, z and $x + dx$, $y + dy$, $z + dz$, are the coordinates of the extremities of the radii, the projections of the area upon the planes of xy, xz, yz, are respectively equal to

$$x\,dy - y\,dx, \quad x\,dz - z\,dx, \quad y\,dz - z\,dy:$$

wherefore, according to a well known property, we shall have,

$$\frac{r^4 dv^2}{\mu\, dt^2} = \frac{(x\,dy - y\,dx)^2}{\mu\, dt^2} + \frac{(x\,dz - z\,dx)^2}{\mu\, dt^2} + \frac{(y\,dz - z\,dy)^2}{\mu\, dt^2}:$$

and the differential of this equation, dt being constant, may be thus written,

$$d \,.\, \frac{r^4 dv^2}{\mu\, dt^2} = 2\,(x^2 + y^2 + z^2) \,.\, \left(\frac{dx\,ddx + dy\,ddy + dz\,ddz}{\mu\, dt^2}\right)$$

$$- 2\,(x\,dx + y\,dy + z\,dz) \,.\, \left(\frac{x\,ddx + y\,ddy + z\,ddz}{\mu\, dt^2}\right).$$

Now, substitute the values of the second differentials taken from the equations (A), and we shall obtain, first,

$$\frac{dx\,ddx + dy\,ddy + dz\,ddz}{\mu\, dt^2} = \frac{d\mathrm{R}}{dx}\,dx + \frac{d\mathrm{R}}{dy}\,dy + \frac{d\mathrm{R}}{dz}\,dz - \frac{dr}{r^2} = d'\mathrm{R} - \frac{dr}{r^2}:$$

and, secondly,

$$\frac{x\,ddx + y\,ddy + z\,ddz}{\mu\, dt^2} = \frac{d\mathrm{R}}{dx}\,x + \frac{d\mathrm{R}}{dy}\,y + \frac{d\mathrm{R}}{dz}\,z - \frac{1}{r} = \frac{d\mathrm{R}}{dr}\,r - \frac{1}{r}:$$

wherefore, since $x^2 + y^2 + z^2 = r^2$ and $x\,dx + y\,dy + z\,dz = r\,dr$, the foregoing differential equation will become by substitution,

$$d \,.\, \frac{r^4 dv^2}{\mu\, dt^2} = 2\,r^2 \left(d'\mathrm{R} - \frac{d\mathrm{R}}{dr}\,dr\right),$$

or, which is equivalent,

$$d \,.\, \frac{r^4 dv^2}{\mu\, dt^2} = 2\,r^2 \left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right).$$

By integrating,

$$\left.\begin{aligned} r^2\,dv &= h\,dt\sqrt{\mu},\\ h^2 &= h_0^2 + 2\int r^2\left(d\mathrm{R} - \frac{d\mathrm{R}}{dr}\,dr\right),\\ h^2 &= h_0^2 + 2\int r^2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right),\end{aligned}\right\} \;.\;.\;.\;.\;.\;.\;.\;.\;(3)$$

the constant h_0 being equal to $\frac{r^2\,dv}{dt\sqrt{\mu}}$ when $t = 0$.

Further, let the first of the equations (A) multiplied by y be subtracted from the second multiplied by x; then

$$\frac{d.(x\,dy - y\,dx)}{\mu\,dt^2} = \frac{d\mathrm{R}}{dy}\,x - \frac{d\mathrm{R}}{dx}\,y:$$

and, by converting the quantities in this equation into functions of r, λ, s,

$$\frac{d.\left(\frac{r^2}{1+s^2}\cdot\frac{d\lambda}{dt\sqrt{\mu}}\right)}{dt\sqrt{\mu}} = \frac{d\mathrm{R}}{d\lambda}:$$

and by multiplying both sides by $2\,.\,\frac{r^2}{1+s^2}\,.\,d\lambda$,

$$d.\left(\frac{r^2}{1+s^2}\cdot\frac{d\lambda}{dt\sqrt{\mu}}\right)^2 = 2\,\frac{r^2}{1+s^2}\cdot\frac{d\mathrm{R}}{d\lambda}\,d\lambda:$$

and, by integrating,

$$\left.\begin{aligned} \frac{r^2\,d\lambda}{1+s^2} &= h'\,dt\sqrt{\mu},\\ h'^2 &= h_0'^2 + 2\int\frac{r^2}{1+s^2}\cdot\frac{d\mathrm{R}}{d\lambda}\,d\lambda,\end{aligned}\right\} \;.\;.\;.\;.\;.\;.\;.\;.\;(4)$$

h_0' being a constant.

The equations that have been investigated, which are only three, the first and second being one equation in two different forms, are sufficient for determining the place of a planet at any proposed instant of time, whether it revolves solely by the central force of the sun, or is disturbed by the irregular attractions of the other bodies of the system. The second and third equations ascertain the form and magnitude of the orbit in its proper plane, and the place of the planet; the fourth equation enables us to find the angle in which the plane of the orbit is inclined to the immoveable plane of $x\,y$, and the position of the line in which the two planes intersect one another.

3. We begin with the more simple case of the problem, when the planet is urged solely by the central force of the sun. On this supposition, there being no disturbing forces, we must make $R = 0$ in the equations of the last §. By the formulas (3) and (4), we have,

$$r^2\,d\,v = h\,d\,t\,\sqrt{\mu},$$

$$\frac{r^2}{1+s^2}\,.\,d\lambda = h'\,d\,t\,\sqrt{\mu};$$

and h, h', are constant quantities. Now $\frac{r}{\sqrt{1+s^2}}$ is the projection of r upon the plane of $x\,y$; and the area $\frac{r^2}{1+s^2}\,.\,d\lambda$ is the projection of the area $r^2\,d\,v$ upon the same plane; wherefore, if i denote the angle of inclination which the plane containing the radii vectores r and $r + d\,r$, has to the plane of $x\,y$, we shall have

$$\cos i = \frac{\frac{r^2}{1+s^2}\,.\,d\lambda}{r^2\,d\,v} = \frac{h'}{h};$$

which proves that a plane passing through the sun's centre and any two places of the planet infinitely near one another, has constantly the same inclination to the immoveable plane of $x\,y$. And it further proves that the planet moves in one invariable plane; for, unless this were the case, the areas described round the sun in any consecutive small portions of time, could not constantly have the same proportion to their projections upon the plane of $x\,y$.

The orbit in its proper plane will be determined by the equations (2) and (3), viz.

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{a} = 0,$$

$$r^2\,d\,v = h\,d\,t\,\sqrt{\mu},$$

a and h being arbitrary quantities. By exterminating $d\,t\,\sqrt{\mu}$ from the first equation,

$$h^2\,.\,\frac{d\,r^2}{r^4\,d\,v^2} + \frac{h^2}{r} - \frac{2}{r} + \frac{1}{a} = 0;$$

by multiplying all the terms by $\frac{r^2}{a}$, and adding 1 to both sides,

$$\frac{h^2}{a}\cdot\frac{d\,r^2}{r^2\,d\,v^2}+\left(1-\frac{r}{a}\right)^2=1-\frac{h^2}{a};$$

and by introducing the new quantity e^2,

$$e^2=1-\frac{h^2}{a}$$

$$(1-e^2)\frac{d\,r^2}{r^2\,d\,v^2}+\left(1-\frac{r}{a}\right)^2=e^2.$$

This equation is solved by assuming

$$\frac{d\,r}{r\,d\,v}=\frac{e\sin\theta}{1+e\cos\theta},$$

$$1-\frac{r}{a}=e\times\frac{\cos\theta+e}{1+e\cos\theta},$$

the arc θ remaining indeterminate. For, if the assumed quantities be substituted, the equation will be verified, and the arc θ will be eliminated. In order to determine θ, let the second of the formulas be differentiated, and equate $\frac{dr}{r}$ to the like value in the first formula; then,

$$d\,v=d\,\theta;\ \text{and}\ v-\varpi=\theta.$$

The nature of the orbit is therefore determined by these two equations,

$$\frac{d\,r}{r\,d\,v}=\frac{e\sin(v-\varpi)}{1+e\cos(v-\varpi)},$$

$$r=\frac{a\,(1-e^2)}{1+e\cos(v-\varpi)}:$$

the first of which shows that the two conditions $\frac{d\,r}{d\,v}=0$, and $\sin(v-\varpi)=0$, must take place at the same time; so that ϖ is the place of the planet when its distance from the sun is a minimum $=a\,(1-e)$, or a maximum $=a\,(1+e)$: and the second proves that the orbit of the planet is an ellipse having the sun in one focus; a being the mean distance; e the eccentricity; and $v-\varpi$ the true anomaly, that is, the angular distance from the perihelion or aphelion;

from the perihelion if e be positive, and from the aphelion if the same quantity be negative.

It must however be observed that the preceding determination rests entirely on the assumption that, in the equation $e^2 = 1 - \frac{h^2}{a}$, the quantity $\frac{h^2}{a}$ is positive and less than unit. Without entering upon any detail, which our present purpose does not require, all the possible cases of the problem will be succinctly distinguished by writing the equation in this form,

$$\frac{h^2}{1+e} = (1-e) \times a.$$

The quantity on the left side being essentially positive, the two factors on the other side must both have the same sign. If they are positive, the orbit will be an ellipse; if they are negative, and consequently e greater than unit, the curve described by the body will be a hyperbola; and it will be a parabola, when $e = 1$, and a and $1-e$ pass from being positive to be negative, at which limit the equation will assume this form,

$$\frac{h^2}{2} = 0 \times \infty.$$

In all the cases $\frac{h^2}{1+e}$ is the perihelion distance.

The nature of the orbit being found, we have next to determine the relation between the time and the angular motion of the planet. For this purpose we have the equation, $r^2\,dv = h\,dt\sqrt{\mu}$, from which, by substituting the values of r and h, we deduce

$$\frac{dt\sqrt{\mu}}{a^{\frac{3}{2}}} = \frac{(1-e^2)^{\frac{3}{2}}\,dv}{\big(1+e\cos(v-\varpi)\big)^2}.$$

Let $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}} = n$; then, by integrating,

$$nt + \epsilon - \varpi = \int \frac{(1-e^2)^{\frac{3}{2}}\,dv}{\big(1+e\cos(v-\varpi)\big)^2},$$

the quantity under the sign of integration being taken so as to vanish when $v - \varpi = 0$, and ϵ being a constant quantity. The mean motion of the planet reckoned from a given epoch, is equal to $nt + \epsilon$; and the mean anomaly, to

$nt+\epsilon-\varpi$, the true anomaly being $v-\varpi$. The equation may be put in this form,

$$nt+\epsilon-\varpi=\int\frac{\sqrt{1-e^2}\,.\,dv}{1+e\cos(v-\varpi)}-e\times\frac{\sqrt{1-e^2}\,.\,\sin(v-\varpi)}{1+e\cos(v-\varpi)}:$$

and, if we assume

$$\sin u=\frac{\sqrt{1-e^2}\,.\,\sin(v-\varpi)}{1+e\cos(v-\varpi)},\ \cos u=\frac{\cos(v-\varpi)+e}{1+e\cos(v-\varpi)};$$

we shall find,

$$u=\int\frac{\sqrt{1-e^2}\,.\,dv}{1+e\cos(v-\varpi)}:$$

so that we readily arrive at these results,

$$nt+\epsilon-\varpi=u-e\sin u,$$

$$r=\frac{a(1-e^2)}{1+e\cos(v-\varpi)}=a(1-e\cos u),$$

$$\tan\frac{v-\varpi}{2}=\tan\frac{u}{2}\times\sqrt{\frac{1+e}{1-e}}.$$

These last are the formulas that occur in the solution of KEPLER's problem, the arc u being the anomaly of the eccentric. Having found the expression of the eccentric anomaly in terms of the mean anomaly from the first of the formulas, we thence deduce the true anomaly $v-\varpi$, and the radius vector r, for any proposed instant of time. The analytical solution of these questions is omitted; the sole intention of treating here of the motion of a planet circulating by the central force of the sun, being to elucidate the investigations that are to follow respecting the orbit of a disturbed planet.

The purposes of astronomy require further that the motion of the planet in its orbit be connected with the longitudes and latitudes estimated with regard to the immovable plane of xy. The orbit being supposed to intersect the immovable plane, and the angle of inclination being represented by i, let N stand for the longitude of the ascending node, and P for the place of the same node in the plane of the orbit and reckoned from the same origin with the true motion v: then $v-\text{P}$, or the distance of the planet from the node in the plane of the orbit, is the hypothenuse of a right-angled spherical triangle, one side of which is the arc $\lambda - \mathrm{N}$ in the immovable plane, and the remaining side is the latitude having s for its tangent: wherefore we have

$$\tan(\lambda - \mathrm{N}) = \tan(v - \mathrm{P}) \cos i,$$

$$s = \tan i \sin(\lambda - \mathrm{N}).$$

The first of these equations enables us to compute λ when v is given, and conversely; by means of the second, the latitude is found. The practical calculations are much facilitated by expressing the quantities sought in converging serieses: but the discussion of these points is beside our present purpose.

4. We now proceed to investigate the effect of the disturbing force of the planet P′ in altering the orbit of P. For this purpose we have the equations (3) and (4), viz.

$$r^2\, dv = h\, dt \sqrt{\mu},$$

$$\frac{r^2}{1+s^2} \cdot d\lambda = h'\, dt \sqrt{\mu};$$

of which the first is the expression of the small area described round the sun by the planet in the time dt, and the other is the projection of that area upon the immovable plane of xy. Wherefore, if i denote the angle of inclination which the plane passing through the sun and the radii vectores r and $r + dr$, has to the plane of xy, we shall have

$$\cos i = \frac{\frac{r^2}{1+s^2} \cdot d\lambda}{r^2\, dv} = \frac{h'}{h}:$$

and, as h' and h vary incessantly by the action of the disturbing forces, it follows that the momentary plane in which the planet moves is continually changing its inclination to the fixed plane. Let i' be the value of i when $t = 0$; then $\cos i' = \frac{h'_0}{h_0}$; and, by the formulas (3) and (4), we shall have,

$$h^2 = h_0^2 + 2\int r^2 \left(\frac{d\mathrm{R}}{d\lambda}\, d\lambda + \frac{d\mathrm{R}}{ds}\, ds\right),$$

$$h'^2 = h_0^2 \cos^2 i' + 2\int r^2 \cdot \frac{d\mathrm{R}}{d\lambda} \cdot \frac{d\lambda}{1+s^2},$$

$$h''^2 = h^2 - h'^2 = h_0^2 \sin^2 i' + 2\int r^2 \left(\frac{d\mathrm{R}}{d\lambda} \cdot \frac{s^2\, d\lambda}{1+s^2} + \frac{d\mathrm{R}}{ds}\, ds\right):$$

and hence, in consequence of what has been shown,

$$\cos^2 i = \frac{h'^2}{h^2}\,;\ \sin^2 i = \frac{h''^2}{h^2}\,;\ \tan^2 i = \frac{h''^2}{h'^2}.$$

Let the momentary plane of the planet's orbit, that is, the plane passing through r and $r + dr$, intersect the immovable plane of xy, and put N for the place of the ascending node: then s and $s + ds$ will be the tangents of the latitudes at the distances $\lambda - \text{N}$, and $\lambda + d\lambda - \text{N}$ from the node: and, i being the angle contained between the two planes, we shall have,

$$\left.\begin{aligned} s &= \tan i \sin(\lambda - \text{N}), \\ \frac{ds}{d\lambda} &= \tan i \cos(\lambda - \text{N}). \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots \quad (5)$$

By adding the squares of these equations,

$$s^2 + \frac{ds^2}{d\lambda^2} = \tan^2 i = \frac{h''^2}{h'^2}\,;$$

by differentiating, making $d\lambda$ constant,

$$h'^2\left(\frac{dds}{d\lambda^2} + s\right) = \frac{h''\,dh'' - h'\,dh'\left(s^2 + \frac{ds^2}{d\lambda^2}\right)}{ds}\,;$$

and, by substituting the values of $h''\,dh''$ and $h'\,dh'$,

$$\frac{dds}{d\lambda^2} + s = \frac{r^3}{h'^2}\cdot\left\{\frac{d\text{R}}{ds} - \frac{1}{1+s^2}\cdot\frac{d\text{R}}{d\lambda}\cdot\frac{ds}{d\lambda}\right\}.$$

Since i is variable in the equations (5), it is obvious that N, or the place of the node, must likewise vary. By combining each of the two equations with the differential of the other, these results will be obtained,

$$0 = \frac{d.\tan i}{d\lambda}\sin(\lambda - \text{N}) - \frac{d\text{N}}{d\lambda}\tan i \cos(\lambda - \text{N})$$

$$\frac{dds}{d\lambda^2} + s = \frac{d.\tan i}{d\lambda}\cos(\lambda - \text{N}) + \frac{d\text{N}}{d\lambda}\tan i \sin(\lambda - \text{N})\,;$$

from which we deduce,

$$di = \cos^2 i \cos(\lambda - \text{N}).\left\{\frac{dds}{d\lambda^2} + s\right\}.d\lambda,$$

$$d\text{N} = \frac{\cos i \sin(\lambda - \text{N})}{\sin i}.\left\{\frac{dds}{d\lambda^2} + s\right\}.d\lambda\,;$$

and, by substituting the value of $\frac{d\,d\,s}{d\,\lambda^2} + s$,

and, observing that $r^2\,d\,\lambda = (1 + s^2)\,h'\,d\,t\,\sqrt{\mu} = (1 + s^2)\,h \cos i\,d\,t\,\sqrt{\mu}$, we finally get,

$$\left.\begin{aligned} d\,i &= \cos i \cos(\lambda - \mathrm{N}) \,.\, \left\{(1 + s^2)\frac{d\,\mathrm{R}}{d\,s} - \frac{d\,\mathrm{R}}{d\,\lambda} \cdot \frac{d\,s}{d\,\lambda}\right\} \cdot \frac{d\,t\,\sqrt{\mu}}{h} \\ d\,\mathrm{N} &= \frac{\sin(\lambda - \mathrm{N})}{\sin i} \cdot \left\{(1 + s^2)\frac{d\,\mathrm{R}}{d\,s} - \frac{d\,\mathrm{R}}{d\,\lambda} \cdot \frac{d\,s}{d\,\lambda}\right\} \cdot \frac{d\,t\,\sqrt{\mu}}{h} \end{aligned}\right\} \quad . \; . \quad (6)$$

These equations determine the motion of the node in longitude, and the variation in the inclination of the orbit. They are rigorously exact, and may be transformed in various ways, as it may suit the purpose of the inquirer.

We proceed now to investigate the motion of the planet round the sun. For this purpose we have the equations (2) and (3), viz.

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{a} = 2\int d'\,\mathrm{R},$$

$$r^2\,d\,v = h\,d\,t\,\sqrt{\mu}.$$

And first, as the small arc $d\,v$ contained between the two radii r and $r + d\,r$, continually passes from one plane to another, it is requisite to inquire what notion we must affix to the sum v. The momentary plane of the planet's motion, in shifting its place, turns upon a radius vector; and if we suppose a circle concentric with the sun to be described in it, and to remain firmly attached to it, the differentials $d\,v$ will evidently accumulate upon the circumference of the circle, and will form a continuous sum, in the same manner as if the plane remained motionless in one position. The arc v is therefore the angular motion of the planet round the sun in the moveable plane, and is reckoned upon the circumference of the circle from an arbitrary origin.

In the first of the foregoing equations a is an arbitrary constant, and I shall put,

$$\frac{1}{\mathrm{a}} = \frac{1}{a} - 2\int d'\,\mathrm{R};$$

so that we shall have

$$\frac{d\,r^2}{\mu\,d\,t^2} + \frac{r^2\,d\,v^2}{\mu\,d\,t^2} - \frac{2}{r} + \frac{1}{\mathrm{a}} = 0,$$

$$r^2\,d\,v = h\,d\,t\,\sqrt{\mu};$$

which are different from the corresponding equations in the last section in no respect, except that here h and a are both variable, whereas in the other case they were both constant. Treating these equations exactly as before, we first get by exterminating $dt\sqrt{\mu}$,

$$h^2 \frac{dr^2}{r^4\,dv^2} + \frac{h^2}{r^2} - \frac{2}{r} + \frac{1}{\mathrm{a}} = 0;$$

then, by multiplying all the terms by $\frac{r^2}{\mathrm{a}}$ and adding 1 to both sides,

$$\frac{h^2}{\mathrm{a}} \cdot \frac{dr^2}{r^2\,dv^2} + \left(1 - \frac{r}{\mathrm{a}}\right)^2 = 1 - \frac{h^2}{\mathrm{a}};$$

from which we deduce

$$e^2 = 1 - \frac{h^2}{\mathrm{a}},$$

$$(1-e^2)\,\frac{dr^2}{r^2\,dv^2} + \left(1 - \frac{r}{\mathrm{a}}\right)^2 = e^2.$$

The last equation is solved by the same assumptions as before, viz.

$$\frac{dr}{r\,dv} = \frac{e\sin\theta}{1 + e\cos\theta},$$

$$1 - \frac{r}{\mathrm{a}} = e \times \frac{\cos\theta + e}{1 + e\cos\theta};$$

but it must be recollected that in these formulas, a and e are both variable. By differentiating the expression of r, viz.

$$r = \frac{\mathrm{a}\,(1 - e^2)}{1 + e\cos\theta} = \frac{h^2}{1 + e\cos\theta},$$

we get,

$$\frac{dr}{r} = \frac{e\sin\theta\,.\,d\theta}{1 + e\cos\theta} + \frac{2\,dh}{h} - \frac{\cos\theta\,.\,de}{1 + e\cos\theta};$$

and by equating this expression to the value of $\frac{dr}{r}$ taken from the first formula, and reducing, we obtain,

$$e\,(dv - d\theta)\sin\theta + \cos\theta\,.\,de = \frac{2\,dh}{h}\,(1 + e\cos\theta).$$

It appears therefore that $v - \theta$, or ϖ, is a variable quantity; and the formulas that determine the elliptic orbit, and the variation of ϖ, are as follows:

$$\frac{d\,r}{r\,d\,v} = \frac{e\sin(v - \varpi)}{1 + e\cos(v - \varpi)}$$

$$r = \frac{\mathrm{a}(1 - e^2)}{1 + e\cos(v - \varpi)} = \frac{h^2}{1 + e\cos(v - \varpi)},$$

$$e\,d\varpi\sin(v - \varpi) + d\,e\,.\cos(v - \varpi) = \frac{2\,d\,h}{h}\Big(1 + e\cos(v - \varpi)\Big). \;.\;.\;.\; (7)$$

It is obvious that this last formula is tantamount to the equating to zero of the differential of r relatively to the variables, $h, e, \varpi,$ or a, e, ϖ; it may therefore be thus written,

$$\frac{d\,r}{d\,\mathrm{a}}d\,\mathrm{a} + \frac{d\,r}{d\,e}d\,e + \frac{d\,r}{d\,\varpi}d\,\varpi = 0. \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\; (8)$$

The equations that have been investigated, enable us to deduce from the disturbing forces the variable elements of the ellipse that coincides momentarily with the real path of the planet; a being the mean distance, e the eccentricity; ϖ the place of the perihelion, and h^2 the semi-parameter. We have next to find the relation between the time and the angular motion in the variable orbit. This will be accomplished by means of the equation $r^2\,d\,v = h\,d\,t\sqrt{\mu}$; from which we obtain, by substituting the values of r and h,

$$\frac{d\,t\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}} = \frac{(1 - e^2)^{\frac{3}{2}}\,.\,d\,v}{\Big(1 + e\cos(v - \varpi)\Big)^2}.$$

The integral $\int\frac{d\,t\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}}$, supposed to commence with the time, is the mean motion of the planet: when there is no disturbing force, a being constant, the mean motion is proportional to the time and equal to $n \times t$; but the action of the disturbing forces, by making a variable, alters the case, and requires the introducing of a new symbol ζ to represent the mean motion. Thus we have

$$\zeta = \int\frac{\mu}{\mathrm{a}^{\frac{3}{2}}} \times d\,t; \quad d\zeta = \frac{(1 - e^2)^{\frac{3}{2}}\,.\,d\,v}{\Big(1 + e\cos(v - \varpi)\Big)^2}.$$

The value of ζ cannot be obtained directly by integration on account of the variability of e and ϖ. Let $f(v - \varpi, e)$ express that function of the true anomaly which is equal to the mean anomaly in the undisturbed orbit; that is, suppose,

$$f(v-\varpi, e) = \int \frac{(1-e^2)^{\frac{3}{2}} \,.\, dv}{\left(1 + e\cos(v-\varpi)\right)^2},$$

the integral being taken on the conditions that it vanishes when $v - \varpi = 0$, and that e and ϖ are constant. If now we make e and ϖ variable, we shall have,

$$\frac{d\,.f(v-\varpi, e)}{dv}\,dv + \frac{d\,.f(v-\varpi, e)}{de}\,de + \frac{d\,.f(v-\varpi, e)}{d\varpi}\,d\varpi = d\,.f(v-\varpi, e).$$

But the partial differential relatively to v, is no other than the expression of $d\zeta$: wherefore,

$$d\zeta + \frac{d\,.f(v-\varpi, e)}{de}\,de + \frac{d\,.f(v-\varpi, e)}{d\varpi}\,d\varpi = d\,.f(v-\varpi, e).$$

By introducing a new symbol this equation may be separated into the two which follow,

$$d\zeta + d\epsilon - d\varpi = d\,.f(v-\varpi, e),$$

$$d\epsilon - d\varpi = \frac{d\,.f(v-\varpi, e)}{de}\,de + \frac{d\,.f(v-\varpi, e)}{d\varpi}\,d\varpi.$$

In the integral

$$\zeta + \epsilon - \varpi = f(v-\varpi, e),$$

$\zeta + \epsilon$ is the mean motion of the planet reckoned from a given epoch, ϵ however representing a quantity that varies incessantly by the action of the disturbing forces, the amount of the variation being determined by the second formula in which the value of ϵ alone has not been previously ascertained. The mean anomaly of the planet is $\zeta + \epsilon - \varpi$; and the integral shows that there is the same finite equation between the mean and the true anomalies in the disturbed orbit as when there is no disturbing force. It follows therefore that, in both the cases, the true anomaly, the true motion of the planet, and the radius vector, are deducible from the mean anomaly by the same rules and by the solution of KEPLER's problem.

In order to find the value of the new variable ϵ, it is necessary to eliminate the differential coefficients from its expression. Differentiating relatively to e and ϖ, we shall get,

$$\frac{d\,.f(v-\varpi, e)}{de} = -\sqrt{1-e^2}\,.\int \frac{2\cos(v-\varpi) + 3e + e^2\cos(v-\varpi)}{\left(1 + e\cos(v-\varpi)\right)^3}\,.\,dv,$$

$$\frac{d.f(v-\varpi, e)}{d\varpi} = -(1-e^2)^{\frac{1}{2}} . \int \frac{2e\sin(v-\varpi) . dv}{\left(1+e\cos(v-\varpi)\right)^3}:$$

and, by integrating,

$$\frac{d.f(v-\varpi, e)}{de} = -\frac{\sin(v-\varpi)\sqrt{1-e^2}}{1+e\cos(v-\varpi)} - \frac{\sin(v-\varpi)\sqrt{1-e^2}}{\left(1+e\cos(v-\varpi)\right)^2} =$$

$$-\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)\sqrt{1-e^2}}{\left(1+e\cos(v-\varpi)\right)^2},$$

$$\frac{d.f(v-\varpi, e)}{d\varpi} = -\frac{(1-e^2)^{\frac{3}{2}}}{\left(1+e\cos(v-\varpi)\right)^2}.$$

These values being substituted in the foregoing formula, we shall find this result, after dividing all the terms by the coefficient of $d\varpi$,

$$\frac{\left(1+e\cos(v-\varpi)\right)^2}{(1-e^2)^{\frac{3}{2}}} . (d\epsilon - d\varpi) = -\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)}{1-e^2} de - d\varpi,$$

or, more concisely,

$$\frac{a^2\sqrt{1-e^2}}{r^2} . (d\epsilon - d\varpi) = -\frac{\left(2+e\cos(v-\varpi)\right)\sin(v-\varpi)}{1-e^2} . de - d\varpi. . \quad . \quad . \ (9)$$

From the equation between the mean and the true anomalies we deduce,

$$v = \zeta + \epsilon - \Phi,$$

Φ representing a function of the mean anomaly $\zeta + \epsilon - \varpi$: and as the differentials of ζ and v are independent of the differentials of ϵ, e, and ϖ, we shall have,

$$\frac{dv}{d\epsilon}d\epsilon + \frac{dv}{de}de + \frac{dv}{d\varpi}d\varpi = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \ (10)$$

Now,

$$\frac{dv}{d\epsilon} = 1 - \frac{d.\Phi}{d\epsilon}; \quad \frac{dv}{d\varpi} = -\frac{d.\Phi}{d\varpi}:$$

and, because, Φ is a function of $\epsilon - \varpi$,

$$\frac{d.\Phi}{d\epsilon} = -\frac{d.\Phi}{d\varpi}: \quad \text{consequently, } \frac{dv}{d\epsilon} + \frac{dv}{d\varpi} = 1.$$

The equation may therefore be thus written,

$$\frac{dv}{d\varepsilon}(d\varepsilon - d\varpi) + \frac{dv}{de}\,de + d\varpi = 0.$$

But, v being a function $\zeta + \varepsilon$, it follows that,

$$\frac{dv}{d\varepsilon} = \frac{dv}{d\zeta} = \frac{a^2\sqrt{1-e^2}}{r^2};$$

and thus it appears that the equation we are considering is identical with the formula (9): from which we learn that,

$$\frac{dv}{de} = \frac{\big(2 + e\cos(v-\varpi)\big)\sin(v-\varpi)}{1-e^2}.$$

It remains now to say a word about the longitudes and latitudes of the planet reckoned on the immoveable plane of xy. The variable quantities N and i denote the longitude of the ascending node, and the inclination of the orbit, in respect to the fixed plane: let P represent the place of the same node on the moveable plane of the planet, this arc being reckoned from the same origin as the true motion v: then, because the momentary plane in which the planet moves, in taking a new position, turns about a radius vector, it is obvious that, if dN be the motion of the node in the fixed plane of xy, $\cos i \times d\text{N}$ will be its motion in the variable plane of the orbit. Wherefore we have,

$$d\text{P} = \cos i \times d\text{N}, \text{ and } \text{P} = \int \cos i \,.\, d\text{N},$$

a constant being supposed to accompany the integral. This being observed, it is obvious that the same equations as in the case of the undisturbed orbit, will obtain between the quantities under consideration, viz.

$$\tan(\lambda - \text{N}) = \cos i \tan(v - \text{P}),$$
$$s = \tan i \sin(\lambda - \text{N}).$$

The foregoing investigations prove that the motion of a disturbed planet may be accurately represented by a variable ellipse coinciding momentarily with the real path of the planet. The variations, in the magnitude, the form, and the position of the ellipse, have been expressed by equations that depend upon the disturbing forces. A new inquiry presents itself: to exhibit the differentials of the elements of the variable orbit in the forms best adapted for use.

5. The expressions of the coordinates x, y, z, in terms of the variables r, λ, s, are as follows:

$$x = \frac{r\cos\lambda}{\sqrt{1+s^2}}, \quad y = \frac{r\sin\lambda}{\sqrt{1+s^2}}, \quad z = \frac{rs}{\sqrt{1+s^2}}:$$

and, if we write $\lambda - \mathrm{N} + \mathrm{N}$ for λ, we shall get,

$$x = r \,.\left\{\frac{\cos(\lambda-\mathrm{N})}{\sqrt{1+s^2}}\cos\mathrm{N} - \frac{\sin(\lambda-\mathrm{N})}{\sqrt{1+s^2}}\sin\mathrm{N}\right\},$$

$$y = r \,.\left\{\frac{\sin(\lambda-\mathrm{N})}{\sqrt{1+s^2}}\cos\mathrm{N} + \frac{\cos(\lambda-\mathrm{N})}{\sqrt{1+s^2}}\sin\mathrm{N}\right\}.$$

But $v - \mathrm{P}$ in the plane of the planet's motion is the hypothenuse of a right-angled spherical triangle of which $\lambda - \mathrm{N}$ is one side, s the tangent of the other side, and i the angle opposite to this latter side; and from these considerations we get

$$\frac{\cos(\lambda-\mathrm{N})}{\sqrt{1+s^2}} = \cos(v-\mathrm{P}), \quad \frac{\sin(\lambda-\mathrm{N})}{\sqrt{1+s^2}} = \sin(v-\mathrm{P})\cos i, \text{ and}$$

$$\frac{s}{\sqrt{1+s^2}} = \sin(v-\mathrm{P})\sin i:$$

wherefore we have these values of the coordinates,

$$x = r\,.\{\cos(v-\mathrm{P})\cos\mathrm{N} - \sin(v-\mathrm{P})\sin\mathrm{N}\cos i\}$$

$$y = r\,.\{\sin(v-\mathrm{P})\cos\mathrm{N}\cos i + \cos(v-\mathrm{P})\sin\mathrm{N}\}$$

$$z = r\,.\sin(v-\mathrm{P})\sin i.$$

The radius vector r is a function of v, a, e, ϖ, viz.

$$r = \frac{\mathrm{a}(1-e^2)}{1+e\cos(v-\varpi)}:$$

and thus the coordinates x, y, z, are functions of v and the five elements a, e, ϖ, N, i; for P is no independent quantity, since it varies with N. In order to abridge we may write X, Y, Z for the multipliers of r in the foregoing expressions of x, y, z; so that

$$x = r \times \mathrm{X}, \quad y = r \times \mathrm{Y}, \quad z = r \times \mathrm{Z}.$$

Now, on account of the equation (8) we have

$$\frac{dx}{d\mathrm{a}}d\mathrm{a}+\frac{dx}{de}de+\frac{dx}{d\varpi}d\varpi=\left\{\frac{dr}{d\mathrm{a}}d\mathrm{a}+\frac{dr}{de}de+\frac{dr}{d\varpi}d\varpi\right\}\times \mathrm{X}=0\,;$$

and, in like manner,

$$\frac{dy}{d\mathrm{a}}d\mathrm{a}+\frac{dy}{de}de+\frac{dz}{d\varpi}d\varpi=0,$$

$$\frac{dz}{d\mathrm{a}}d\mathrm{a}+\frac{dz}{de}de+\frac{dz}{d\varpi}d\varpi=0.$$

Further, we have,

$$\frac{dx}{d\mathrm{N}}d\mathrm{N}+\frac{dx}{di}di=r\,.\left\{\left(\frac{d\mathrm{X}}{d\mathrm{P}}\cos i+\frac{d\mathrm{X}}{d\mathrm{N}}\right)d\mathrm{N}+\frac{d\mathrm{X}}{di}\,di\right\};$$

and, if the expression on the right side of this formula be computed, it will be found equal to

$$\{\sin(v-\mathrm{P})\,di-\cos(v-\mathrm{P})\sin i\,d\mathrm{N}\}\times\sin\mathrm{N}\sin i\,;$$

and, by substituting the values of sin (*v* – P) and cos (*v* – P), the same quantity may be thus written,

$$\{\sin(\lambda-\mathrm{N})\,d\,.\tan i-\cos(\lambda-\mathrm{N})\tan i\,d\mathrm{N}\}\times\frac{\sin\mathrm{N}\sin i\cos i}{\sqrt{1+s^2}}\,;$$

which expression is equal to zero in consequence of what was shown in § 4. Wherefore we have,

$$\left.\begin{aligned}&\frac{dx}{d\mathrm{N}}d\mathrm{N}+\frac{dx}{di}di=0\,;\\&\text{and similarly,}\\&\frac{dy}{d\mathrm{N}}d\mathrm{N}+\frac{dy}{di}di=0\\&\frac{dz}{d\mathrm{N}}d\mathrm{N}+\frac{dz}{di}di=0.\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ (11)$$

It follows from what has been said that the expressions of dx, dy, dz contain dv only, and are independent of the differentials of the five elements, a, e, ϖ, N, i, which destroy one another and disappear. And further, if in x, y, z we substitute for v, its value in terms of the mean motion and the mean anomaly, viz.

$$v=\zeta+\epsilon-\Phi,$$

the expressions of dx, dy, dz will contain $d\zeta$ only: for dv contains $d\zeta$ only, and is independent of the differentials of ι, e, ϖ. Thus we have

$$dx = \frac{dx}{d\zeta}\,d\zeta = \frac{dx}{dt}\,dt, \quad dy = \frac{dy}{d\zeta}\,d\zeta = \frac{dy}{dt}\,dt, \quad dz = \frac{dz}{d\zeta}\,d\zeta = \frac{dz}{dt}\,dt.$$

It is in these properties that we recognise the principle of the *Variation of the arbitrary constants.* The finite expressions of x, y, z, being the same in the immoveable ellipse described by the sole action of the centripetal force of the sun, and in the variable ellipse which represents the motion of a disturbed planet, they will verify the equations (A), supposing the arbitrary quantities constant, and neglecting the disturbing forces. The velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ are the same whether the arbitrary quantities remain constant or vary; and thus, for a moment of time dt, the motion in the invariable ellipse coincides with that of the planet in its real path. But, in the next moment of time, the planet will quit the periphery of the ellipse supposed to continue invariable; because the forces in that orbit are different from the forces which urge the planet. In the immoveable ellipse the forces in the directions of the coordinates are equal to $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$, the arbitrary quantities being constant; but, in the case of the planet, the like forces are equal to the same differentials augmented by the variation of the arbitrary quantities, the additions thus introduced being equal to the disturbing forces, $\mu\frac{d\mathrm{R}}{dx}$, $\mu\frac{d\mathrm{R}}{dy}$, $\mu\frac{d\mathrm{R}}{dz}$. It is in this manner that an elliptic orbit, by the variation of its elements, is capable of representing at every moment of time both the velocity of a disturbed planet, and the forces by which it is urged.

And generally, when a dynamical problem admits of an exact solution, the arbitrary quantities may be made to vary so as not to alter the velocities $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$; and the additions which the variation of the same quantities makes to the expressions $\frac{ddx}{dt^2}$, $\frac{ddy}{dt^2}$, $\frac{ddz}{dt^2}$ will represent new forces introduced in the problem. By means of this artifice we may estimate the effect of any disturbing forces, more especially of such as bear an inconsiderable proportion to the principal forces, in altering the original motion of the body. This is the principle of the *Variation of the arbitrary constants*, a method which has been much discussed, and which is now probably exhausted. It originated in the first researches on physical astronomy, and has been matured in passing through the hands of EULER, LAGRANGE, LAPLACE, and POISSON. The labours of these great geometers have raised up a general analytical theory applicable to every case, and requiring no more than the substitution of the particular forces under consideration. Invaluable as are such extensive views, the application of formulas constructed on considerations of so general a nature, may not always be very ready or very direct, and may require much subordinate calculation. In important problems it may be advantageous to separate the principles of the method from the analytical processes with which they are conjoined, and to deduce the solution directly from the principles themselves by attending closely to the peculiar nature of the case.

Distinguishing the two planets by their masses m and m', the symbol R stands for a function of x, y, z, the coordinates of the disturbed planet m, and of x', y', z', those of the disturbing planet m'. The expressions of these latter coordinates will be obtained by marking all the quantities in the values of x, y, z, with an accent, understanding that the accented quantities denote the same things relatively to the orbit of m', that the unaccented quantities represent in the orbit of m. The function R may be transformed in two ways, according as we substitute, for the coordinates, one set of values or another. It will be changed into a function of the four independent quantities r, v, N, i, and of the like four accented quantities of the planet m', by substituting the values of the coordinates obtained in the beginning of this section: and in this case, for greater precision, the partial differentials of R relatively to r and v will be written with parentheses, thus, $\left(\frac{d\mathrm{R}}{dr}\right)$ and $\left(\frac{d\mathrm{R}}{dv}\right)$. When the values of x, y, z, in terms of the mean motion ζ and of the six elements, a, ϵ, e, ϖ, N, i, and the like values of the other coordinates are substituted, R will be a function of the mean motions ζ and ζ', and of the respective elements of the two orbits. In this latter transformation, the partial differentials of R will be written, as usual, without parentheses. It may not be improper to set down here the expressions of such of these partial differentials as we shall have occasion to refer to,

$$\frac{d\mathrm{R}}{d\mathrm{a}}=\left(\frac{d\mathrm{R}}{dr}\right)\cdot\frac{dr}{d\mathrm{a}}=\left(\frac{d\mathrm{R}}{dr}\right)\cdot\frac{r}{\mathrm{a}},$$

$$\frac{d\mathrm{R}}{d\epsilon}=\left(\frac{d\mathrm{R}}{dr}\right)\cdot\frac{dr}{dv}\cdot\frac{dv}{d\epsilon}+\left(\frac{d\mathrm{R}}{dv}\right)\cdot\frac{dv}{d\epsilon}, \qquad \text{(C)}$$

$$\frac{d\mathrm{R}}{de}=\left(\frac{d\mathrm{R}}{dr}\right)\cdot\left(\frac{dr}{de}+\frac{dr}{dv}\cdot\frac{dv}{de}\right)+\left(\frac{d\mathrm{R}}{dv}\right)\cdot\frac{dv}{de},$$

$$\frac{d\mathrm{R}}{d\varpi}=\left(\frac{d\mathrm{R}}{dr}\right)\cdot\left(\frac{dr}{d\varpi}+\frac{dr}{dv}\cdot\frac{dv}{d\varpi}\right)+\left(\frac{d\mathrm{R}}{dv}\right)\cdot\frac{dv}{d\varpi},$$

in which expressions, it need hardly be observed, that $\frac{dv}{d\epsilon}$, $\frac{dv}{de}$, $\frac{dv}{d\varpi}$, refer to this value of v,

$$v=\zeta+\epsilon-\Phi.$$

Proceeding now to reduce the differentials of the elements of the variable orbit to the forms best adapted for use, we have this formula for the mean distance a,

$$\frac{1}{\mathrm{a}}=\frac{1}{a}-2\int d'\mathrm{R}: \text{ consequently, } \frac{d\mathrm{a}}{\mathrm{a}^2}=2\,d'\mathrm{R}.$$

Now, when x, y, z are transformed into expressions of ζ and the elements of the orbit, it has been proved that dx, dy, dz contain $d\zeta$ only, and are independent of the differentials of the elements; wherefore, the value $d'\mathrm{R}$ will be found by differentiating R, making ζ the only variable, that is, we shall have,

$$d'\mathrm{R}=\frac{d\mathrm{R}}{dx}dx+\frac{d\mathrm{R}}{dy}dy+\frac{d\mathrm{R}}{dz}dz=\frac{d\mathrm{R}}{d\zeta}d\zeta.$$

But substituting this value,

$$\left.\begin{aligned} d\mathrm{a}&=2\mathrm{a}^2\,\frac{d\mathrm{R}}{d\zeta}\,d\zeta,\\ \mathrm{a}&=a+2\int\mathrm{a}^2\,\frac{d\mathrm{R}}{d\zeta}\,d\zeta, \end{aligned}\right\}\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \qquad (12)$$

The mean motion ζ is defined by this equation, $d\zeta=\frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}}$. But, we have,

$$\frac{1}{a}=\frac{1}{\mathrm{a}}\left(1+2\mathrm{a}\int d'\mathrm{R}\right);\text{ and, }\frac{dt\sqrt{\mu}}{a^{\frac{3}{2}}}=\frac{dt\sqrt{\mu}}{\mathrm{a}^{\frac{3}{2}}}\cdot\left(1+2\mathrm{a}\int d'\mathrm{R}\right)^{\frac{3}{2}}.$$

Let $n^2 = \frac{\mu}{a^3}$, n being the constant of the mean motion in the primitive ellipse, when $t = 0$: then

$$\left.\begin{aligned} n\,dt &= d\zeta\left(1 + 2\mathrm{a}\int\frac{d\mathrm{R}}{d\zeta}d\zeta\right)^{\frac{3}{2}} \\ d\zeta &= n\,dt - d\zeta\left\{3\mathrm{a}\int\frac{d\mathrm{R}}{d\zeta}d\zeta + \frac{3}{2}\mathrm{a}^2\left(\int\frac{d\mathrm{R}}{d\zeta}d\zeta\right)^2 \text{ \&c.}\right. \end{aligned}\right\}\ldots (13)$$

Taking next the semi-parameter h^2, we have, by equation (3),

$$h\,dh = r^2\left(d'\mathrm{R} - \frac{d\mathrm{R}}{dr}dr\right):$$

but $d'\mathrm{R} = \left(\frac{d\mathrm{R}}{dv}\right)dv + \left(\frac{d\mathrm{R}}{dr}\right)dr$; wherefore,

$$h\,dh = \left(\frac{d\mathrm{R}}{dv}\right).r^2\,dv = \mathrm{a}^2\sqrt{1-e^2}\,.\left(\frac{d\mathrm{R}}{dv}\right)d\zeta.$$

In order to find the value of $\left(\frac{d\mathrm{R}}{dv}\right)$, let the expressions of $\frac{d\mathrm{R}}{d\epsilon}$ and $\frac{d\mathrm{R}}{d\varpi}$ in the formulas (C), be added: then, since it has been shown that $\frac{dv}{d\epsilon} + \frac{dv}{d\varpi} = 1$, we get,

$$\frac{d\mathrm{R}}{d\epsilon} + \frac{d\mathrm{R}}{d\varpi} = \left(\frac{d\mathrm{R}}{dr}\right)\left(\frac{dr}{d\varpi} + \frac{dr}{dv}\right) + \left(\frac{d\mathrm{R}}{dv}\right):$$

and, because r is a function of $v - \varpi$, $\frac{dr}{dv} + \frac{dr}{d\varpi} = 0$; wherefore,

$$\frac{d\mathrm{R}}{d\epsilon} + \frac{d\mathrm{R}}{d\varpi} = \left(\frac{d\mathrm{R}}{dv}\right).$$

Further, because ϵ always accompanies ζ, or which is the same thing, because R is a function of $\zeta + \epsilon$, we have $\frac{d\mathrm{R}}{d\zeta} = \frac{d\mathrm{R}}{d\epsilon}$: consequently,

$$\frac{d\mathrm{R}}{d\zeta} + \frac{d\mathrm{R}}{d\varpi} = \left(\frac{d\mathrm{R}}{dv}\right).$$

By substituting this value,

$$\left.\begin{aligned} h\,dh &= \mathrm{a}^2\sqrt{1-e^2}\,.\left(\frac{d\mathrm{R}}{d\zeta} + \frac{d\mathrm{R}}{d\varpi}\right)d\zeta, \\ h^2 &= a(1-e^2) + 2\int\mathrm{a}^2\sqrt{1-e^2}\left(\frac{d\mathrm{R}}{d\zeta} + \frac{d\mathrm{R}}{d\varpi}\right)d\zeta, \end{aligned}\right\}\ldots (14)$$

the semi-parameter of the primitive ellipse being equal to $a(1-e'^2)$, and its eccentricity to e'.

The eccentricity is determined by this formula,

$$e^2 = 1 - \frac{h^2}{\mathrm{a}}:$$

by differentiating,

$$e\,de = -\frac{h\,dh}{\mathrm{a}} + \frac{h^2}{2}\cdot\frac{d\,\mathrm{a}}{\mathrm{a}^2} = -\frac{h\,dh}{\mathrm{a}} + \mathrm{a}\,(1-e^2)\frac{d\,\mathrm{R}}{d\,\zeta}\,d\,\zeta:$$

and by substituting the value of $h\,dh$,

$$de = -\mathrm{a}\sqrt{1-e^2}\,.\left\{\frac{1-\sqrt{1-e^2}}{e}\cdot\frac{d\,\mathrm{R}}{d\,\zeta} + \frac{d\,\mathrm{R}}{e\,d\varpi}\right\}d\,\zeta. \quad . \quad . \quad . \qquad (15)$$

For the variation of the perihelion we have the formula (7), which may be written in this manner,

$$\frac{2}{r}h\,dh = \cos(v-\varpi)\,de + e\sin(v-\varpi)\,d\varpi:$$

and by multiplying all the terms by e,

$$e\sin(v-\varpi)\,.\,e\,d\varpi = \frac{2e}{r}h\,dh - \cos(v-\varpi)\,e\,de:$$

and because $e\,de = -\frac{h\,dh}{\mathrm{a}} + h^2\,d'\mathrm{R}$,

$$e\sin(v-\varpi)\,.\,e\,d\varpi = \left(\frac{2e}{r} + \frac{\cos v-\varpi}{\mathrm{a}}\right)h\,dh - h^2\cos(v-\varpi)\,d'\mathrm{R}.$$

Further, $d'\mathrm{R} = \frac{h\,dh}{r^2} + \left(\frac{d\,\mathrm{R}}{dr}\right)dr = \frac{h\,dh}{r^2} + \frac{1}{r^2}\cdot\frac{dr}{dv}\cdot\left(\frac{d\,\mathrm{R}}{dr}\right).r^2\,dv$:

and, by substituting this value,

$$e\sin(v-\varpi)\,.\,e\,d\varpi = \left(\frac{2e}{r} + \frac{\cos(v-\varpi)}{\mathrm{a}} - \frac{h^2\cos(v-\varpi)}{r^2}\right).\,h\,dh$$

$$-\frac{h^2}{r^2}\cdot\frac{dr}{dv}\cos(v-\varpi)\left(\frac{d\,\mathrm{R}}{dr}\right)r^2\,dv.$$

Now $\frac{h^2}{r^2}\cdot\frac{dr}{dv} = e\sin(v-\varpi)$; and it will be found that the coefficient of $h\,dh$ is equal to,

$$\frac{\big(2+e\cos(v-\varpi)\big).\,e\sin^2(v-\varpi)}{\mathrm{a}\,(1-e^2)} = \frac{1}{\mathrm{a}}\cdot\frac{dv}{de}\cdot e\sin(v-\varpi):$$

wherefore, by substituting and dividing all the terms by $e\sin(v-\varpi)$,

$$e\,d\varpi = \frac{1}{a}\cdot\frac{dv}{de}\cdot h\,dh - \cos(v-\varpi)\left(\frac{dR}{dr}\right).r^2\,dv.$$

But $h\,dh = \left(\frac{dR}{dv}\right)r^2\,dv$, and,

$$-\cos(v-\varpi) = \frac{1}{a}\left(\frac{dr}{de}+\frac{dr}{dv}\cdot\frac{dv}{de}\right):$$

wherefore,

$$e\,d\varpi = \left\{\frac{dv}{de}\left(\frac{dR}{dv}\right) + \left(\frac{dr}{de}+\frac{dr}{dv}\cdot\frac{dv}{de}\right)\left(\frac{dR}{dr}\right)\right\}.\frac{r^2\,dv}{a}:$$

and, because $r^2\,dv = a^2\sqrt{1-e^2}:d\zeta$,

$$d\varpi = \frac{a\sqrt{1-e^2}}{e}.\frac{dR}{de}.d\zeta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The variation of ϵ, the longitude of the epoch, must be deduced from the equation (9), viz.

$$\frac{a^2\sqrt{1-e^2}}{r^2}(d\epsilon - d\varpi) = -\frac{dv}{de}de - d\varpi.$$

From this $d\,e$ may be eliminated by means of the equation (7), viz.

$$\frac{2}{r}\,h\,dh = \cos(v-\varpi)\,de + e\sin(v-\varpi)\,d\varpi;$$

and the result will be

$$\frac{a^2\sqrt{1-e^2}.\cos(v-\varpi)}{r^2}(d\epsilon - d\varpi) =$$

$$-\frac{2}{r}\cdot\frac{dv}{de}\,h\,dh - \left\{\cos(v-\varpi) - e\sin(v-\varpi)\frac{dv}{de}\right\}d\varpi.$$

Now the coefficient of $d\varpi$ is equal to

$$(1-e^2)\frac{a^2}{r^2}\cos(v-\varpi) - \frac{2ae}{r};$$

wherefore, by multiplying by r^2, we get

$$a^2\sqrt{1-e^2}.\cos(v-\varpi)(d\epsilon - d\varpi) = -2r.\frac{dv}{de}.h\,dh$$

$$-a\{a(1-e^2)\cos(v-\varpi) - 2re\}\,d\varpi;$$

substitute now the value of $d\varpi$ in equation (16), and that of $h\,dh$ viz. $\mathrm{a}^2\sqrt{1-e^2}\left(\frac{d\mathrm{R}}{dv}\right)d\zeta$, then

$$\cos(v-\varpi)\,(d\epsilon-d\varpi)=-2r\frac{dv}{de}\cdot\left(\frac{d\mathrm{R}}{dv}\right)d\zeta$$

$$-\left\{\frac{\mathrm{a}(1-e^2)\cos v-\varpi}{e}-2r\right\}\frac{d\mathrm{R}}{de}d\zeta.$$

But, as appears from the formulas (C),

$$\frac{dv}{de}\cdot\left(\frac{d\mathrm{R}}{dv}\right)=\frac{d\mathrm{R}}{d\epsilon}+\mathrm{a}\cos(v-\varpi)\left(\frac{d\mathrm{R}}{dv}\right):$$

wherefore, by substituting and dividing all the terms by $\cos(v-\varpi)$,

$$d\epsilon-d\varpi=-\frac{\mathrm{a}(1-e^2)}{e}\cdot\frac{d\mathrm{R}}{de}d\zeta-2r\mathrm{a}\left(\frac{d\mathrm{R}}{dr}\right)d\zeta:$$

and by substituting the value of $d\varpi$, and observing that $\left(\frac{d\mathrm{R}}{dr}\right)=\frac{d\mathrm{R}}{d\mathrm{a}}\cdot\frac{\mathrm{a}}{r}$, we obtain

$$d\epsilon=\mathrm{a}\sqrt{1-e^2}\left(\frac{1-\sqrt{1-e^2}}{e}\right)\frac{d\mathrm{R}}{de}d\zeta-2\,\mathrm{a}^2\frac{d\mathrm{R}}{d\mathrm{a}}.d\zeta.\quad.\quad.\quad.\quad.\quad(17)$$

If the formulas (C) be multiplied, each by its own differential, and the respective results be added, it will be found that the coefficients of $\left(\frac{d\mathrm{R}}{dr}\right)$ and $\left(\frac{d\mathrm{R}}{dv}\right)$ are each equal to zero, on account of the equations (8) and (10): so that we have,

$$\frac{d\mathrm{R}}{d\mathrm{a}}d\mathrm{a}+\frac{d\mathrm{R}}{de}de+\frac{d\mathrm{R}}{d\epsilon}d\epsilon+\frac{d\mathrm{R}}{d\varpi}d\varpi=0:$$

and this equation will serve to verify the values of $d\mathrm{a}$, de, $d\epsilon$, $d\varpi$, which have been separately investigated.

It remains to examine whether the values of di and $d\mathrm{N}$ already found (equation (6)), can be expressed similarly to the other elements. The three quantities N, P, i, or rather the two N and i, since P varies with N, are independent of r and v, and consequently of ζ, a, e, ϵ, ϖ: wherefore, by differentiating the expressions of x, y, z relatively to i, we shall get

$$\frac{dx}{di}=r\sin(v-\mathrm{P})\sin\mathrm{N}\sin i=z\sin\mathrm{N},$$

$$\frac{dy}{di} = -r\sin(v-P)\cos N\sin i = -z\cos N,$$

$$\frac{dz}{di} = r\sin(v-P)\cos i = \frac{r\sin(\lambda-N)}{\sqrt{1+s^2}}.$$

Let these expressions be multiplied respectively by $\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$, and then added; the result will be

$$\frac{dR}{di} = z\left\{\frac{dR}{dx}\sin N - \frac{dR}{dy}\cos N\right\} + \frac{dR}{dz}\cdot\frac{r\sin(\lambda-N)}{\sqrt{1+s^2}}:$$

and by substituting the values contained in the formulas (B),

$$\frac{dR}{di} = (1+s^2)\frac{dR}{ds}\sin(\lambda-N) - \frac{dR}{d\lambda}s\cos(\lambda-N):$$

and, because $s\cos(\lambda-N) = \frac{ds}{d\lambda}\sin(\lambda-N)$,

$$\frac{dR}{di} = \sin(\lambda-N)\cdot\left\{(1+s^2)\frac{dR}{ds} - \frac{dR}{d\lambda}\cdot\frac{ds}{d\lambda}\right\}.$$

If the equations (11) be multiplied respectively by $\frac{dR}{dx}$, $\frac{dR}{dy}$, $\frac{dR}{dz}$, and then added, this result will be obtained,

$$\frac{dR}{di}di + \frac{dR}{dN}dN = 0.$$

By combining this equation and the value of $\frac{dR}{di}$ with the formulas (6), we get,

$$\left.\begin{aligned} dN &= \frac{a}{\sqrt{1-e^2}}\cdot\frac{1}{\sin i}\cdot\frac{dR}{di}\cdot d\zeta, \\ di &= -\frac{a}{\sqrt{1-e^2}}\cdot\frac{1}{\sin i}\cdot\frac{dR}{dN}\cdot d\zeta. \end{aligned}\right\} \quad \dots\dots\dots\dots \quad (18)$$

The differentials of the several elements of the orbit of the disturbed planet have now been made to depend upon the function R and its differentials relatively to the elements themselves and to the mean motion ζ. Upon the calculations which this transformation requires, which have long ago been carried as far as human perseverance can well be supposed to go, we do not here enter. The variations of the elements of a disturbed planet, in the most perfect form in which they have been exhibited in the latter part of this paper, are the result of the repeated labours of Lagrange and Laplace, who, at different times and by different methods, at last succeeded in overcoming the difficulties of this great problem.

In this paper the utmost rigour of investigation has been strictly preserved. No admission or supposition has any where been made for the sake of simplifying calculation or of obtaining a result more readily. The procedure that has been followed likewise makes it easy to change the form of the differentials of the elements of the orbit, as occasion may require. Thus it is obvious from the formulas (C), and from other formulas, that the variations of all the elements may be expressed by means of the three functions $\frac{d\mathrm{R}}{dr}$, $\frac{d\mathrm{R}}{d\lambda}$, $\frac{d\mathrm{R}}{ds}$; or, by means of the three $\frac{d\mathrm{R}}{dx}$, $\frac{d\mathrm{R}}{dy}$, $\frac{d\mathrm{R}}{dz}$; or, by any two of the differentials of R relatively to a, e, ϵ, ϖ, and one of the two, relatively to i and N; which remark is useful in the theory of the comets.

There is this advantage in expressing the differentials of the elements by means of the function R, that inspection alone discovers the nature of the terms that enter into every formula. But it is not enough to know the form of the terms, we must likewise attend to their convergency. In the present state of the heavens there is no difficulty in this respect, because the eccentricities of the planetary orbits, and their inclinations to the ecliptic are found to be small, and it is upon the smallness of these quantities that the convergency of the series into which R is developed, mainly depends. In the present circumstances of the planetary system, the formulas afford the utmost possible facility for computing the inequalities of the elliptic elements. After all, the inquiry is difficult enough when it is carried beyond a first approximation; for in the second stage of the process every element that enters into a formula being itself a collection of sines or cosines, it is not easy to be assured of the nature of the quantities arising from the combination of so many complex expressions.

If we extend our views and consider the stability of the system of the world, it is necessary to begin with establishing the convergency of the terms into which R is expanded. The mathematical form of these terms will always be the same; but unless their total amount can be estimated with sufficient exactness by a limited number of them, the human understanding can come to no solid decision. Now this will depend upon the effect of the perturbations in changing the eccentricities and the inclinations of the orbits to the ecliptic.

If it can be proved that these elements, after an indefinite lapse of time, will remain of inconsiderable magnitude as they are at present, the convergency of the series will be established, and the form of the terms of which R consists, will enable us to compute the changes in all the elliptic elements, and to decide the great question of the stability of the system. But we cannot enter upon any extended discussion of these points, and shall conclude this paper with some remarks in illustration of the problem we have solved, and of the manner in which we have solved it.

6. If we suppose that there is no disturbing force, or that $\mathrm{R} = 0$, we shall have by the equation (1),

$$\frac{1}{a} = \frac{2}{r} - \frac{dx^2 + dy^2 + dz^2}{dt^2 \cdot \mu};$$

and if V represent the velocity of the planet at the extremity of r, then,

$$\mathrm{V}^2 = \frac{dx^2 + dy^2 + dz^2}{dt^2 \cdot \mu}, \text{ and } \frac{1}{a} = \frac{2}{r} - \mathrm{V}^2.$$

This last equation shows that the mean distance a of an elliptic orbit depends only upon the radius vector drawn to any point, and upon the velocity at that point. Conceive that the straight line r extends from the sun in a given direction and to a given length, and from its extremity suppose that a planet is launched into space with the velocity V, the foregoing equation will determine the mean distance a of the immoveable ellipse in which the planet will revolve. The point from which the planet is projected, and consequently r, remaining the same, $\frac{1}{a}$ and V^2 will vary together; and if we suppose that a becomes equal to a, at the same time that V^2 is changed into $\mathrm{V}^2 + d \cdot \mathrm{V}^2$ by forces which act continually but insensibly, we shall have these equations,

$$d \cdot \frac{1}{\mathrm{a}} = - d \cdot \mathrm{V}^2, \quad \text{and } \frac{1}{\mathrm{a}} = \frac{1}{a} - \int d \cdot \mathrm{V}^2.$$

It has been shown that the disturbing forces acting in the directions of x, y, z, and tending to increase these lines, are respectively $\frac{d\mathrm{R}}{dx}$, $\frac{d\mathrm{R}}{dy}$, $\frac{d\mathrm{R}}{dz}$: and, by the principles of dynamics, double the sum of the products of these forces, each being multiplied by the element of its direction, is equal to the change effected on the square of the velocity: wherefore,

$$2\left(\frac{d\mathrm{R}}{dx}\,dx + \frac{d\mathrm{R}}{dy}\,dy + \frac{d\mathrm{R}}{dz}\,dz\right) = 2\,d'\mathrm{R} = d \cdot \mathrm{V}^2;$$

and consequently,

$$d\,.\,\frac{1}{\mathrm{a}} = -\,2\,d'\mathrm{R}, \qquad \text{and}\ \frac{1}{\mathrm{a}} = \frac{1}{a} - 2\int d'\mathrm{R}.$$

These results agree with the investigation in the fourth section of this paper; and they coincide with the remarkable equation first discovered by Lagrange, from which he inferred the invariability of the mean distances and the periodic times of the planets, when the approximation is extended to the first power only of the disturbing force.

It has already been observed that $\frac{d\mathrm{R}}{d\lambda}\,.\,\frac{\sqrt{1+s^2}}{r}$ is the disturbing force perpendicular to the plane passing through the sun and the coordinate z, that is, to the planet's circle of latitude; and likewise that $\frac{d\mathrm{R}}{ds}\,.\,\frac{1+s^2}{r}$ is the disturbing force in the same plane perpendicular to r the radius vector. The elements of the direction of these forces are respectively $\frac{r\,d\lambda}{\sqrt{1+s^2}}$ and $\frac{r\,ds}{1+s^2}$: wherefore,

$$2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right)$$

is the variation produced in the square of the velocity in the direction perpendicular to r. But dv being the small angle described round the sun in the time dt, the space described by the planet perpendicular to r, is $r\,dv$; and consequently $\frac{r\,dv}{dt\,\sqrt{\mu}}$ is the planet's velocity in that direction. Wherefore, using the symbol δ to denote a variation caused by the disturbing forces perpendicular to the radius vector, and observing that these forces produce no momentary increase or decrease of that line, we get,

$$2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right) = \delta\,.\,\frac{r^2\,dv^2}{dt^2\,.\,\mu} = r^2\,\delta\,.\,\frac{dv^2}{dt^2\,.\,\mu}:$$

consequently,

$$2\,r^2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right) = r^4\,\delta\,.\,\frac{dv^2}{dt^2\,.\,\mu} = \delta\,.\,\frac{r^4\,dv^2}{dt^2\,.\,\mu}:$$

and, as $\frac{r^2\,dv}{dt\sqrt{\mu}}$ and its square vary by no other cause but the action of the forces perpendicular to r, we have

$$2\,r^2\left(\frac{d\mathrm{R}}{d\lambda}\,d\lambda + \frac{d\mathrm{R}}{ds}\,ds\right) = d\,.\,\frac{r^4\,\mathrm{d}v^2}{\mathrm{d}t^2\,.\,\mu}.$$

Now this is the same differential equation that has already been obtained by a different method in equation (3) of the second section, and from which the value of h^2, the semi-parameter of the variable elliptic orbit, was deduced. That element is therefore as much an immediate deduction from the disturbing forces, as is the mean distance in the equation of LAGRANGE. As the variation of a is the effect of the disturbing force in altering the velocity in the orbit, so the variation of h^2 is the effect of that part of the disturbing force which alters the exact proportionality to the times of the areas described round the sun. The two elements are together sufficient for determining both the form and the magnitude of the momentary elliptic orbit. The placing of this ellipse so as to be in intimate contact with the real path of the planet, a procedure which corresponds to finding the relation between the arcs θ and v, determines the motion of the line of the apsides.

If, lastly, we attend to that part of the disturbing force which is perpendicular to the circle of latitude passing through the planet, and proceed as before, we shall obtain the differential of the equation (4) in the second section. This differential is therefore the effect of the disturbing force in altering the momentary area which is described in the immoveable plane of xy, and which, without the action of this force would be proportional to the time. The elementary area in the immoveable plane is the projection of the area described in the same time in the plane of the orbit; the proportion of the two determines the cosine of the inclination of the variable plane in which the planet moves; and from this it is easy to determine the position of the line of the nodes, as has been fully explained.

What has been said is independent of the nature of the forces in action; and it is obvious that the same method may be applied to estimate the effect of any extraneous force in disturbing the elliptic motion of a planet.

It would appear that in the view we have taken of this problem, we have been making an approach to some general hints contained in the corollaries of the seventeenth proposition of the first book of the Principia. A connexion between the most recondite results of modern analytical science, and the original ideas thrown out by an author who, although he accomplished so much, has unavoidably left much to be supplied by his successors, is undoubtedly worthy of being remarked, and may suggest useful reflections.

December 22, 1831.

VIII.—*Researches in Physical Astronomy.* *By* J. W. Lubbock, *Esq. V. P. and Treas. R.S.*

Read February 9, 1832.

In general, two methods present themselves of solving any mechanical problem: the one furnished by the variation of parameters or constants, which complete the integral obtained by the first approximation; the other furnished by the integration of the differential equations by means of indeterminate coefficients, or some equivalent method. Each of these methods may be applied to the theory of the perturbations of the heavenly bodies; and they lead to expressions which are, of course, substantially identical, but which do not appear in the same shape except after certain transformations.

My object in the following pages is to effect these transformations, by which their identity is established, making use of the developments of R and $r\frac{dR}{dr}$ given in the Philosophical Transactions for 1831, p. 295. The identity of the results obtained by either method serves to confirm the exactness of those expressions.

Integrating the equation

$$\frac{d^2 r^2}{2\,d t^2} - \frac{\mu}{r} + \frac{\mu}{a} + 2\int d R + r\left(\frac{d R}{d r}\right) = 0$$

omitting the terms which are independent of the quantities b, and which result from the part of R which is equal to $\frac{r\,r_{,}\cos(\lambda - \lambda_{,})}{r_{,}^3}$, and the factor $\frac{m_{,}}{\mu}$.

$$\frac{a}{r} = \frac{n^2}{\{i(n-n_{,})+n\}\{i(n-n_{,})-n\}}\left\{\frac{n\,a}{(n-n_{,})\,a_{,}}\,b_{1,i} + \frac{a}{2\,a_{,}}\,\frac{d\,b_{1,i}}{d\,a}\right\}\cos i\,(n\,t - n_{,}t)$$

$$+ \frac{2\,(i+1)\,n^3}{\big(i\,(n-n_{,})+n\big)\big(i\,(n-n_{,})+2\,n\big)\,i\,(n-n_{,})}\left\{\frac{a^2}{4\,a_{,}^2}\,b_{3,i-1} + \frac{a^3}{2\,a_{,}^3}\,b_{3,i}\right.$$

$$\left. - \frac{3\,a^2}{4\,a_{,}^2}\,b_{3,i+1}\right\} e\cos\big(i\,(n\,t - n_{,}t) + n\,t - \varpi\big)$$

$$+\frac{n^2}{\big(i(n-n_{,})+2n\big)\,i(n-n_{,})}\left\{\frac{ia^2}{4a_{,}^2}b_{3,i-1}-\frac{(1+2i)}{2}\frac{a^3}{a_{,}^3}b_{3,i}\right.$$

$$\left.+\frac{3ia^2}{4a_{,}^2}b_{3,i+1}\right\}e\cos\big(i(nt-n_{,}t)+nt-\varpi\big)$$

$$+\frac{3\{i(n-n_{,})+n\}^2 n}{2\big(i(n-n_{,})+2n\big)\,i(n-n_{,})\big(i(n-n_{,})-n\big)}\left\{\frac{n}{(n-n_{,})}\frac{a}{a_{,}}b_{1,i}\right.$$

$$\left.+\frac{a}{2a_{,}}\frac{\mathrm{d}\,b_{1,i}}{\mathrm{d}\,a}\right\}e\cos\big(i(nt-n_{,}t)+nt-\varpi\big)$$

$$+\frac{2in^3}{\big(i(n-n_{,})+n_{,}\big)\big(i(n-n_{,})+n+n_{,}\big)\big(i(n-n_{,})-n+n_{,}\big)}\left\{-\frac{3a^2}{4a_{,}^2}b_{3,i-1}+\frac{a}{2a_{,}}b_{3,i}\right.$$

$$\left.+\frac{a^2}{4a_{,}^2}b_{3,i+1}\right\}e_{,}\cos\big(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\big)$$

$$+\frac{n^2}{\big(i(n-n_{,})+n+n_{,}\big)\big(i(n-n_{,})-n+n_{,}\big)}\left\{\frac{3(1+i)}{4}\frac{a^2}{a_{,}^2}b_{3,i-1}-\frac{ia}{a_{,}}b_{3,i}\right.$$

$$\left.-\frac{(1-i)}{4}\frac{a^2}{a_{,}^2}b_{3,i+1}\right\}e_{,}\cos\big(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\big)$$

i being any whole number positive or negative, but excepting the arguments, 0, $nt-\varpi$, $n_{,}t-\varpi_{,}$.

$$\frac{n^3}{(n-n_{,})\big(i(n-n_{,})+n\big)\big(i(n-n_{,})-n\big)}=\frac{n}{n-n_{,}}\left\{\frac{n}{2\big(i(n-n_{,})-n\big)}-\frac{n}{2\big(i(n-n_{,})+n\big)}\right\}$$

$$=\frac{ni}{2\big(i(n-n_{,})-n\big)}-\frac{n}{n-n_{,}}+\frac{ni}{2\big(i(n-n_{,})+n\big)}$$

Resolving the other fractions in the same manner,

$$\frac{a}{r}=-\frac{na}{(n-n_{,})^2a_{,}}b_{1,i}\cos i(nt-n_{,}t)$$

$$-\frac{n}{2\big(i(n-n_{,})+n\big)}\left\{-\frac{2ia}{a_{,}}b_{1,i}-\frac{a^3}{a_{,}^3}b_{3,i}+\frac{a^2}{2a_{,}^2}b_{3,i-1}+\frac{a^2}{2a_{,}^2}b_{3,i+1}\right\}\cos i(nt-n_{,}t)$$

$$+\frac{2(i+1)n}{\{i(n-n_{,})+n\}}\left\{-\frac{a^2}{4a_{,}^2}b_{3,i-1}-\frac{a^3}{2a_{,}^3}b_{3,i}+\frac{3a^2}{4a_{,}^2}b_{3,i+1}\right\}e\cos\big(i(nt-n_{,}t)+nt-\varpi\big)$$

$$+\frac{ne}{2\{i(n-n_{,})+2n\}}\left\{(2i+2)\left\{\frac{a^2}{4a_{,}^2}b_{3,i-1}+\frac{a^3}{2a_{,}^3}b_{3,i}-\frac{3a^2}{4a_{,}^2}b_{3,i+1}\right\}-\frac{ia^2}{4a_{,}^2}b_{3,i-1}\right.$$

$$\left.+\frac{(1+2i)}{2}\frac{a^3}{a_{,}^3}\ {}_{3,i}-\frac{3ia^2}{4a_{,}^2}b_{3,i+1}\right\}\cos\big(i(nt-n_{,}t)+nt-\varpi\big)$$

$$+\frac{ne}{2i(n-n_{,})}\left[(2i+2)\left\{\frac{a^2}{4a_{,}^2}b_{3,i-1}+\frac{a^3}{2a_{,}^3}b_{3,i}-\frac{3a^2}{4a_{,}^2}b_{3,i+1}\right\}+\frac{ia^2}{4a_{,}^2}b_{3,i}\right.$$

$$\left.-\frac{(1+2i)}{2}\frac{a^3}{a_{,}^3}b_{3,i}+\frac{3ia^2}{4a_{,}^2}b_{3,i+1}\right\}\cos\Big(i(nt-n_{,}t)+nt-\varpi\Big)$$

$$+\left\{-\frac{3n^2a}{2i(n-n_{,})^2a_{,}}b_{1,i}+\frac{n}{\Big(i(n-n_{,})-n\Big)}\left\{\frac{2a}{a_{,}}b_{1,i}+\frac{a}{a_{,}}\frac{\mathrm{d}b_{1,i}}{\mathrm{d}a}\right\}\right.$$

$$-\frac{3n}{4i(n-n_{,})}\left\{\frac{3ib_{1,i}}{a_{,}}+\frac{a}{a_{,}}\frac{\mathrm{d}b_{1,i}}{\mathrm{d}a}\right\}$$

$$\left.+\frac{n}{4\Big(i(n-n_{,})+2n\Big)}\left\{\frac{iab_{1,i}}{a_{,}}-\frac{a}{a_{,}}\frac{\mathrm{d}b_{1,i}}{\mathrm{d}a}\right\}\right\}e\cos\Big(i(nt-n_{,}t)+nt-\varpi\Big)$$

$$+\frac{2in}{\Big(i(n-n_{,})+n_{,}\Big)}\left\{\frac{3a^2}{4a_{,}^2}b_{3,i-1}-\frac{a}{2a_{,}}b_{3,i}-\frac{a^2}{4a_{,}^2}b_{3,i+1}\right\}e_{,}\cos\Big(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\Big)$$

$$+\frac{n}{2\Big(i(n-n_{,})-n+n_{,}\Big)}\left\{2i\left\{-\frac{3a^2}{4a_{,}^2}b_{3,i-1}+\frac{a}{2a_{,}}b_{3,i}+\frac{a^2}{2a_{,}^2}b_{3,i+1}\right\}\right.$$

$$\left.-\frac{3(1+i)}{4}\frac{a^2}{a_{,}^2}b_{3,i-1}+\frac{ia}{a_{,}}b_{3,i}+\frac{(1-i)}{4}\frac{a^2}{a_{,}^2}b_{3,i+1}\right\}e_{,}\cos\Big(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\Big)$$

$$+\frac{n}{2\Big(i(n-n_{,})-n+n_{,}\Big)}\left\{2i\left\{-\frac{3a^2}{4a_{,}^2}b_{3,i-1}+\frac{a}{2a_{,}}b_{3,i}+\frac{a^2}{4a_{,}^2}b_{3,i+1}\right\}\right.$$

$$\left.+\frac{3(1+i)}{4}\frac{a^2}{a_{,}^2}b_{3,i-1}-\frac{ia}{a_{,}}b_{3,i}-\frac{(1-i)}{4}\frac{a^2}{a_{,}^2}b_{3,i+1}\right\}e_{,}\cos\Big(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\Big)$$

Observing that

$$ib_{1,i}=\frac{a}{2a_{,}}\left\{b_{3,i-1}-b_{3,i+1}\right\}$$

$$\frac{a\,\mathrm{d}b_{1,i}}{\mathrm{d}a}=-\frac{a}{a_{,}}\left\{\frac{a}{a_{,}}b_{3,i}-\frac{1}{2}b_{3,i-1}-\frac{1}{2}b_{3,i+1}\right\}$$

the preceding expression may be put in the form

$$\frac{a}{r}=-\frac{n}{(n-n_{,})}b_{1,i}\cos i(nt-n_{,}t)$$

$$-\frac{n}{2\{i(n-n_{,})+n\}}\left\{-\frac{a^2}{2a_{,}^2}b_{3,i-1}-\frac{a^3}{a_{,}^3}b_{3,i}+\frac{3a^2}{2a_{,}^2}b_{3,i+1}\right\}\cos i(nt-n_{,}t)$$

$$+\frac{(i+1)n}{\Big(i(n-n_{,})+n\Big)}\left\{-\frac{a^2}{4a_{,}^2}b_{3,i-1}-\frac{a^3}{2a_{,}^3}b_{3,i}+\frac{3a^2}{4a_{,}^2}b_{3,i+1}\right\}e\cos\Big(i(nt-n_{,}t)+nt-\varpi\Big)$$

$$+\frac{ne}{2\left(i(n-n_{,})+2n\right)}\left\{\frac{(i+2)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i-1}+\frac{(3+4i)}{2}\frac{a^{3}}{a_{,}^{3}}b_{3,i}-\frac{(9i+6)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$\cos\left(i(nt-n_{,}t)+nt-\varpi\right)$$

$$+\frac{ne}{2i(n-n_{,})}\left\{(3i+2)\frac{a^{2}}{4a_{,}^{2}}b_{3,i-1}+\frac{a^{3}}{2a_{,}^{3}}b_{3,i}-\frac{(3i+6)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$\cos\left(i(nt-n_{,}t)+nt-\varpi\right)$$

$$+\left\{-\frac{3n^{2}a}{2i(n-n_{,})^{2}a_{,}}b_{1,i}+\frac{n}{\left(i(n-n_{,})-n\right)}\left(\frac{3a^{2}}{2a_{,}^{2}}b_{3,i-1}-\frac{a^{3}}{a_{,}^{3}}b_{3,i}-\frac{a^{3}}{2a_{,}^{3}}b_{3,i+1}\right)\right.$$

$$-\frac{3}{4i(n-n_{,})}\left\{\frac{2a^{2}}{a_{,}^{2}}b_{3,i-1}-\frac{a^{3}}{a_{,}^{3}}b_{3,i}-\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$\left.+\frac{n}{4\left(i(n-n_{,})+2n\right)}\left\{\frac{a^{2}}{a_{,}^{2}}b_{3,i}-\frac{a^{3}}{a_{,}^{3}}b_{3,i+1}\right\}\right\}e\cos\left(i(nt-n_{,}t)+nt-\varpi\right)$$

$$-\frac{ne_{,}}{2\left(i(n-n_{,})+n+n_{,}\right)}\left\{\frac{(3+9i)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i-1}-\frac{2ia}{a_{,}}b_{3,i}-\frac{(1+i)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$\cos\left(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\right)$$

$$-\frac{ne_{,}}{2\left(i(n-n_{,})-n+n_{,}\right)}\left\{\frac{(3i-3)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i-1}-\frac{(3i-1)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$\cos\left(i(nt-n_{,}t)+n_{,}t-\varpi_{,}\right)$$

and by further reductions

$$\frac{a}{r}=\left\{-\frac{nab_{1,i}}{(n-n_{,})a_{,}}-\frac{n}{\left(i(n-n_{,})+n\right)}\left\{-\frac{a^{2}}{4a_{,}^{2}}b_{3,i-1}-\frac{a^{3}}{2a_{,}^{3}}b_{3,i}\right.\right.$$

$$\left.\left.+\frac{3a^{2}}{4a_{,}^{2}}b_{3,i+1}\right\}\right\}\cos i(nt-n_{,}t)$$

$$+\left\{\frac{(i+1)n}{\left(i(n-n_{,})+n\right)}\left\{-\frac{a^{2}}{4a_{,}^{2}}b_{3,i-1}-\frac{a^{3}}{2a_{,}^{3}}b_{3,i}+\frac{3a^{2}}{4a_{,}^{2}}b_{3,i+1}\right\}\right.$$

$$+\frac{n}{\left(i(n-n_{,})+2n\right)}\left\{\frac{(2+i)}{8}\frac{a^{2}}{a_{,}^{2}}b_{3,i-1}+(1+i)\frac{a^{3}}{a_{,}^{3}}b_{3,i}-\frac{(8+9i)}{4}\frac{a^{2}}{a_{,}^{2}}b_{3,i+1}\right\}$$

$$+\frac{n}{i\,(n-n_{,})}\left\{\frac{(3\,i-10)}{8}\,\frac{a^2}{a_{,}^2}\,b_{3,i-1}+\frac{a^3}{a_{,}^3}\,b_{3,i}-\frac{3\,i\,a^2}{8\,a_{,}^2}\,b_{3,i+1}\right\}$$

$$+\frac{n}{\Big(i\,(n-n_{,})-n\Big)}\left\{\frac{3\,a^2}{2\,a_{,}^2}\,b_{3,i-1}-\frac{a^3}{a_{,}^3}\,b_{3,i}-\frac{a^2}{2\,a_{,}^2}\,b_{3,i+1}\right\}$$

$$\left.-\frac{3\,n^2 a}{2\,i\,(n-n_{,})^2}\,\frac{b_{1,i}}{a}\right]e\cos\Big(i\,(n\,t-n_{,}t)+n\,t-\varpi\Big)$$

$$+\left[\frac{n}{\Big(i\,(n-n_{,})+n_{,}\Big)}\left\{\frac{3\,a^2}{4\,a_{,}^2}\,b_{3,i-1}-\frac{a}{2\,a_{,}}\,b_{3,i}-\frac{a^2}{4\,a_{,}^2}\,b_{3,i+1}\right\}\right.$$

$$-\frac{n}{\Big(i\,(n-n_{,})+n+n_{,}\Big)}\left\{\frac{(3+9\,i)}{8}\,\frac{a^2}{a_{,}^2}\,b_{3,i-1}-\frac{i\,a}{a_{,}}\,b_{3,i}-\frac{(1+i)}{8}\,\frac{a^2}{a_{,}^2}\,b_{3,i+1}\right\}$$

$$\left.-\frac{n}{\Big(i\,(n-n_{,})-n+n_{,}\Big)}\left\{\frac{(3\,i-3)}{8}\,\frac{a^2}{a_{,}^2}\,b_{3,i-1}-\frac{(3\,i-1)}{8}\,\frac{a^2}{a_{,}^2}\,b_{3,i+1}\right\}\right]$$

$$e_{,}\cos\Big(i\,(n\,t-n_{,}t)+n_{,}t-\varpi_{,}\Big)$$

i being, as before explained, any whole number positive or negative, excluding only certain arguments, 0, $n\,t+\epsilon-\varpi$, and $n\,t+\epsilon-\varpi_{,}$.

Considering the terms which have hitherto been neglected, if we suppose

$$\frac{a}{r}=1+r_0+e\cos\Big(n\,(1+k)\,t+\epsilon-\varpi\Big)+e_{,}f_{,}\cos\Big(n\,(1+k_{,})\,t+\epsilon-\varpi_{,}\Big),$$

we have $\quad r_0=\frac{a^3}{2\,a_{,}^3}\,b_{3,0}-\frac{a^3}{2\,a_{,}^2}\,b_{3,1},\qquad k=\frac{a^3}{2\,a_{,}^3}\,b_{3,0}-\frac{5\,a^2}{4\,a_{,}^2}\,b_{3,1},\qquad k_{,}=\frac{a^2}{4\,a_{,}^2 f_{,}}\,b_{3,2}.$

See Phil. Trans. 1831, p. 53.

If $n\,(1+2\,r_0)=\mathrm{n}$ and $\mathrm{n}^2=\frac{\mu}{\mathrm{a}^3}$ if e is the coefficient of $\sin\,(n\,t+\epsilon-\varpi)$ in the expression for the longitude, and $f_{,}$ is determined so that the coefficient of $\sin\,(n\,t+\epsilon-\varpi_{,})$ in that expression equals zero,

$$\frac{\mathrm{a}}{r}=1-\frac{a^3}{6\,a_{,}^3}\,b_{3,0}+\frac{a^3}{12\,a_{,}^2}\,b_{3,1}+\mathrm{e}\left\{1+\frac{a^3}{6\,a_{,}^3}\,b_{3,0}-\frac{a^2}{12\,a_{,}^2}\,b_{3,1}\right\}\cos\Big(\mathrm{n}\Big(1-\frac{a^2}{4\,a_{,}^2}\,b_{3,1}\Big)\,t+\epsilon-\varpi\Big)$$

$$+\,e_{,}\left\{\frac{3\,a^2}{8\,a_{,}^2}\,b_{3,0}-\frac{a}{4\,a_{,}}\,b_{3,1}+\frac{a^2}{8\,a_{,}^2}\,b_{3,2}\right\}e_{,}\cos\Big(n\,(1+k_{,})\,t+\epsilon-\varpi_{,}\Big)$$

In the theory of the moon replacing $\frac{m_{,}}{\mu}$,

$$\frac{\mathrm{a}}{r}=1-\frac{m_{,}a^3}{12\,\mu\,a^3}+\mathrm{e}\left\{1+\frac{m_{,}a^3}{12\,\mu\,a_{,}^3}\right\}\cos\Big(\mathrm{n}\Big(1-\frac{3\,m_{,}a^3}{4\,\mu\,a_{,}^3}\Big)\,t+\epsilon-\varpi\Big)$$

$$+\,\frac{3\,m_{,}a^4}{4\,\mu\,a_{,}^4}\,e_{,}\cos\Big(n\,(1+k_{,})\,t+\epsilon-\varpi_{,}\Big).$$

The preceding results are obtained by the direct integration of the differential equations: I shall now show that they coincide with the results obtained by the variation of the elliptic constants.

The equations for determining the variations of the elliptic constants are,

$$\mathrm{d}a = -2a^2n\frac{\mathrm{d}R}{\mathrm{d}\epsilon}\mathrm{d}t$$

$$\mathrm{d}\epsilon = -\frac{an\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t + 2a^2n\frac{\mathrm{d}R}{\mathrm{d}a}\mathrm{d}t$$

$$\mathrm{d}e = \frac{an\sqrt{1-e^2}}{e}(1-\sqrt{1-e^2})\frac{\mathrm{d}R}{\mathrm{d}\epsilon}\mathrm{d}t + \frac{an\sqrt{1-e^2}}{e}\frac{\mathrm{d}R}{\mathrm{d}\varpi}\mathrm{d}t$$

$$\mathrm{d}\varpi = -\frac{an\sqrt{1-e^2}}{e}\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t$$

$$\mathrm{d}\nu = -\frac{an}{\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\iota}\mathrm{d}t$$

$$\mathrm{d}\iota = \frac{an}{\sin\iota\sqrt{1-e^2}}\frac{\mathrm{d}R}{\mathrm{d}\nu}$$

See the Théor. Anal. vol. 1. p. 330, or The Mechanism of the Heavens, p. 231.

In these works R is used with a contrary sign to its acceptation in the Mécanique Céleste, which I have followed.

When the square of the eccentricity is neglected in the value of the radius vector, the equations may be employed in the following shape:

$$\mathrm{d}a = -2a^2n\frac{\mathrm{d}R}{\mathrm{d}\epsilon}\mathrm{d}t \qquad \mathrm{d}\epsilon - \mathrm{d}\varpi = \frac{an}{e}\frac{\mathrm{d}R}{\mathrm{d}e}\mathrm{d}t + 2a^2n\frac{\mathrm{d}R}{\mathrm{d}a}\mathrm{d}t \qquad \mathrm{d}e = \frac{ane}{2}\frac{\mathrm{d}R}{\mathrm{d}\epsilon}\mathrm{d}t + \frac{an}{e}\frac{\mathrm{d}R}{\mathrm{d}\varpi}\mathrm{d}t$$

If $\qquad \zeta = \int n\,\mathrm{d}t \qquad \mathrm{d}\xi = 3\iint an\,\mathrm{d}R\,\mathrm{d}t$

$$\frac{a}{r} = -\frac{\delta a}{a}\left\{1 + e\cos(nt+\epsilon-\varpi)\right\} + \cos(nt+\epsilon-\varpi)\,\delta e - e\sin(nt+\epsilon-\varpi)(\delta\epsilon-\delta\varpi)$$
$$+ 2e\cos(2nt+2\epsilon-2\varpi)\,\delta e - 2e^2\sin(2nt+2\epsilon-2\varpi)(\delta\epsilon-\delta\varpi) - e\sin(nt+\epsilon-\varpi)\,\delta\xi$$

$$\frac{a}{r} = -\frac{na}{(n-n_{\prime})a_{\prime}}b_{1,i}\cos i(nt-n_{\prime}t)$$
$$-\frac{n}{\{i(n-n_{\prime})+n\}}\left\{-\frac{a^2}{4a_{\prime}^2}b_{3,i-1} - \frac{a^3}{2a_{\prime}^3}b_{3,i} + \frac{3a^2}{4a_{\prime}^2}b_{3,i+1}\right\}$$
$$\left\{\cos(nt-\varpi)\cos\left(i(nt-n_{\prime}t)+nt-\varpi\right)\right.$$
$$\left.+\sin(nt-\varpi)\sin\left(i(nt-n_{\prime}t)+nt)-\varpi\right)\right\}$$

$$+\frac{(i+1)\,n}{\{i\,(n-n_{\prime})+n\}}\left\{\frac{a^2}{4\,a_{\prime}^{2}}\,b_{3,i-1}+\frac{a^3}{2\,a_{\prime}^{3}}\,b_{3,i}-\frac{3\,a^2}{4\,a_{\prime}^{2}}\,b_{3,i+1}\right\}\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)$$

$$-\frac{2\,n\,e}{\{i\,(n-n_{\prime})+2\,n\}}\left\{-\frac{(2+i)}{16}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i-1}-\frac{(1+i)}{2}\,\frac{a^3}{a_{\prime}^{3}}\,b_{3,i}+\frac{(8+9\,i)}{16}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i+1}\right\}$$

$$\Big\{\cos\,(n\,t-\varpi)\cos\Big(i\,(n\,t-n_{\prime}\,t)+2\,n\,t-2\,\varpi\Big)$$

$$+\sin\,(n\,t-\varpi)\sin\Big(i\,(n\,t-n_{\prime}\,t)+2\,n\,t-2\,\varpi\Big)\Big\}$$

$$+\frac{n\,e}{i\,(n-n_{\prime})}\left\{\frac{(3\,i-1)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i-1}-\frac{(3\,i+1)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i+1}\right\}\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)$$

$$-\frac{n\,e}{\{i\,(n-n_{\prime})-n\}}\left\{\frac{3\,a^2}{4\,a_{\prime}^{2}}\,b_{3,i-1}-\frac{a^3}{2\,a_{\prime}^{3}}\,b_{3,i}-\frac{a^2}{4\,a_{\prime}^{2}}\,b_{3,i+1}\right\}$$

$$\Big\{\cos\,(2\,n\,t-2\,\varpi)\cos\Big(i\,(n\,t-n_{\prime}\,t)-n\,t+\varpi\Big)$$

$$-\sin\,(2\,n\,t-2\,\varpi)\sin\Big(i\,(n\,t-n_{\prime}\,t)-n\,t+\varpi\Big)$$

$$-\frac{n\,a\,e}{4\,(n-n_{\prime})}\,\frac{b_{1,i}}{a_{\prime}}\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)-\frac{n\,a\,e}{(n-n_{\prime})}\,\frac{b_{1,i}}{a_{\prime}}\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)$$

$$-\frac{a^2\,n\,e}{(n-n_{\prime})}\,\frac{\mathrm{d}\,b_{1,i}}{a_{\prime}\,\mathrm{d}\,a}\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)-\frac{3\,n^2\,a\,e}{2\,i(n-n_{\prime})^2}\,\frac{b_{1,i}}{a_{\prime}}\cos\Big(i(n\,t-n_{\prime}\,t)+n\,t-\varpi\Big)$$

$$-\frac{n\,e_{\prime}}{\{\,i\,(n-n_{\prime})+n+n_{\prime}\,\}}\left\{\frac{(3+9\,i)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i-1}-\frac{i\,a}{a_{\prime}}\,b_{3,i}-\frac{(1+i)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i+1}\right\}$$

$$\Big\{\cos\,(n\,t-\varpi)\cos\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi+n_{\prime}\,t-\varpi_{\prime}\Big)$$

$$-\sin\,(n\,t-\varpi)\sin\Big(i\,(n\,t-n_{\prime}\,t)+n\,t-\varpi+n_{\prime}\,t-\varpi_{\prime}\Big)\Big\}$$

$$-\frac{n\,e_{\prime}}{\{i\,(n-n_{\prime})-n+n_{\prime}\}}\left\{\frac{(3-3\,i)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i-1}-\frac{(1-3\,i)}{8}\,\frac{a^2}{a_{\prime}^{2}}\,b_{3,i+1}\right\}$$

$$\Big\{\cos\,(n\,t-\varpi)\cos\Big(i\,(n\,t-n_{\prime}\,t)-n\,t-\varpi+n_{\prime}\,t-\varpi_{\prime}\Big)$$

$$-\sin\,(n\,t-\varpi)\sin\Big(i\,(n\,t-n_{\prime}\,t)-n\,t-\varpi+n_{\prime}\,t-\varpi_{\prime}\Big)\Big\}$$

It is easily seen that this expression is identical with that of p. 232, obtained by the direct integration of the differential equation of the second order.

Considering the arguments 0, $n\,t+\varepsilon-\varpi$, $n\,t+\varepsilon-\varpi_{\prime}$, still, however, neglecting for an instant the term $\frac{a}{4\,a_{\prime}^{2}}\,b_{3,2}\,e\,e_{\prime}\cos\,(\varpi-\varpi_{\prime})$ which requires particular attention,

$$\frac{a}{r} = \frac{a^3}{2\,a_{,}^3}\,b_{3,0} - \frac{a^2}{2\,a_{,}^2}\,b_{3,1} + \left\{\frac{a^3}{a_{,}^3}\,b_{3,0} - \frac{3\,a^2}{4\,a_{,}^2}\,b_{3,1}\right\} e\cos(n\,t + \epsilon - \varpi)$$

$$+ \left\{\frac{a^3}{a_{,}^3}\,b_{3,0} - \frac{5}{4}\,\frac{a^2}{a_{,}^2}\,b_{3,1}\right\} e\sin(n\,t + \epsilon - \varpi)$$

$$+ \left\{\frac{3}{2}\,\frac{a^2}{a_{,}^2}\,b_{3,0} - \frac{3}{2}\,\frac{a^3}{a_{,}^3}\,b_{3,1} - \frac{a^2}{8\,a_{,}^2}\,b_{3,2}\right\} e_{,}\cos(n\,t + \epsilon - \varpi_{,})$$

$$\frac{a}{r} = \frac{a^3}{2\,a_{,}^3}\,b_{3,0} - \frac{a^2}{2\,a_{,}^2}\,b_{3,1} + \left\{1 + \frac{a^3}{a_{,}^3}\,b_{3,0} - \frac{3}{4}\,\frac{a^2}{a_{,}^2}\,b_{3,1}\right\} e\cos\left(n\left(1 + \frac{a^3}{a_{,}^3}\,b_{3,0}\right.\right.$$

$$\left.\left. - \frac{5}{4}\,\frac{a^2}{a_{,}^2}\,b_{3,1}\right)t + \epsilon - \varpi\right)$$

$$+ \left\{\frac{3}{2}\,\frac{a^2}{a_{,}^2}\,b_{3,0} - \frac{3}{2}\,\frac{a^3}{a_{,}^3}\,b_{3,1} - \frac{a^2}{8\,a_{,}^2}\,b_{3,2}\right\} e_{,}\cos(n\,t + \epsilon - \varpi_{,})$$

The term $\frac{a}{4\,a_{,}^2}\,b_{3,2}\,e\,e_{,}\cos(\varpi - \varpi_{,})$ in the development of R gives

$$\mathrm{d}\,\varpi = -\frac{a^2 n}{4\,a_{,}^2}\,b_{3,2}\,e_{,}\cos(\varpi - \varpi_{,}) \qquad \mathrm{d}\,e = -\frac{a^2 n}{4\,a_{,}^2}\,b_{3,2}\,e_{,}\sin(\varpi - \varpi_{,})$$

If $\quad h = e\sin\varpi \qquad l = e\cos\varpi \qquad h_{,} = e_{,}\sin\varpi_{,} \qquad l_{,} = e_{,}\cos\varpi_{,}$

$$\mathrm{d}\,h = e\cos\varpi\,\mathrm{d}\varpi + \sin\varpi\,\mathrm{d}\,e \qquad = -\frac{a^2 n}{4\,a_{,}^2}\,b_{3,2}\,e_{,}\cos\varpi_{,} = -\frac{a^2\,n}{4\,a_{,}^2}\,b_{3,2}\,l_{,}$$

$$\mathrm{d}\,l = -e\sin\varpi\,\mathrm{d}\,\varpi + \cos\varpi\,\mathrm{d}\,e \qquad = -\frac{a^2 n}{4\,a_{,}^2}\,b_{3,2}\,e_{,}\sin\varpi_{,} = -\frac{a^2\,n}{4\,a_{,}^2}\,b_{3,2}\,h_{,}$$

The integrals of which equations are

$$h = N\sin(g\,t + C) \qquad l = N\cos(g\,t + C)$$

$$h_{,} = N_{,}\sin(g\,t + C) \qquad l = N_{,}\cos(g\,t + C)$$

$$N g = -\frac{a^2 n}{4\,a_{,}^2}\,b_{3,2}\,N_{,} \qquad N_{,}g = -\frac{a\,n_{,}}{4\,a_{,}}\,b_{3,2}\,N$$

$$e\cos(n\,t + \epsilon - \varpi) = N\cos(n - g)\,t + \epsilon - C)$$

which will agree with the previous solution, p. 233, if

$$N = e_{,}f_{,}, \qquad \epsilon - C = \epsilon - \varpi_{,}, \qquad n\,k_{,} = -g.$$

This theory of the secular inequalities appears to require to be extended to the terms depending on higher powers of the eccentricities; but I may remark that the coefficient of the term $e^2\,e_{,}^2\cos(2\,\varpi - 2\,\varpi_{,})$ in the development of R vanishes in the theory of the moon, or at least such part of it as is multiplied by $\frac{a^2}{a_{,}^3}$.

METEOROLOGICAL JOURNAL,

KEPT BY THE ASSISTANT SECRETARY

AT THE APARTMENTS OF THE

ROYAL SOCIETY,

BY ORDER OF

THE PRESIDENT AND COUNCIL.

MDCCCXXXI.

METEOROLOGICAL JOURNAL FOR JULY, 1831.

1831. July.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit.		Self-registering.				
						9 A.M.	3 P.M.	Lowest.	Highest.			
♀ 1	30.116	67.7	30.085	68.4	50	62.7	68.6	56.3	69.2		N	Fine—light clouds and wind.
♄ 2	30.040	67.7	30.007	70.7	47	66.4	72.8	55.8	73.7		WNW	Fine and clear—light clouds.
☉ 3	30.038	69.3	30·050	69.7	60	64.7	70.7	58.4	73.3		SW	A.M. Lowering. P.M. Fine and clear—cloudy.
☽ 4	30.218	68.3	30.235	71.4	49	66.6	73.7	57.7	74.3		W	Fine and clear.
♂ 5	30.238	68.7	30.241	70.3	59	69.3	72.3	59.6	73.3		WSW	A.M. Cloudy—light wind. P.M. Fine and clear. Evening, showery.
☿ 6	30.385	74.6	30.366	73.9	63	71.5	76.7	60.5	80.5		N	Fine and cloudless.
♃ 7	30.365	75.7	30.308	72.7	57	70.3	72.3	56.7	72.7		E	Clear and cloudless.
♀ 8	30.247	64.3	30.225	71.7	59	62.4	72.5	57.3	73.2		NNE	A.M. Cloudy. P.M. Clear and cloudless.
● ♄ 9	30.181	70.2	30.143	76.7	55	71.7	78.7	59.1	80.4		N	Fine and cloudless.
☉ 10	30.060	71.2	30.019	70.3	63	67.6	65.5	65.3	68.7		NNE	Overcast—showery—light brisk wind. Thunder at noon.
☽ 11	29.905	70.5	29.808	71.3	56	67.0	71.6	57.7	72.5	0.069	NNW	Clear—nearly cloudless.
♂ 12	29.665	67.8	29.664	71.8	57	66.0	68.3	62.3	73.3	0.083	WSW	Cloudy—light brisk wind.
☿ 13	29.647	66.4	29.654	66.7	59	61.3	67.5	58.6	67.5	1.014	NW	A.M. Rain. P.M. Fine and clear—cloudy.
♃ 14	29.760	69.3	29.749	68.8	57	63.7	68.4	56.4	68.6	0.094	S	A.M. Lowering—light wind. P.M. Fine and clear.
♀ 15	29.763	67.8	29.783	69.8	59	65.7	68.7	57.3	71.3	0.694	SW	Heavy rain early A.M.—Fine and clear—light clouds.
♄ 16	29·773	67.7	29.797	69.7	60	65.6	68.8	59.5	70.7	0.241	SE	A.M. Clear—cloudy. P.M. Cloudy.
☉ 17	29.963	68.3	30.000	70.2	59	65.0	72.0	57.7	73.4		NW	Fine—light clouds and haze.
☽ 18	30.015	67.7	29.960	70.9	55	68.3	70.9	56.8	72.5		SW	Fine—cloudy.
♂ 19	29.930	69.3	29.918	70.7	53	67.4	68.5	56.7	72.5		SSW	Fair—cloudy.
☿ 20	29.810	67.7	29.809	69.6	55	63.7	65.4	58.8	68.3		SSW	Cloudy.
♃ 21	29.660	69.7	29.659	70.6	61	66.4	70.3	61.7	71.7	0.067	SSW	Fair—cloudy—brisk unsteady wind. Rain and high wind early A.M.
♀ 22	29.845	68.4	29.822	69.3	50	66.7	69.0	52.6	69.7	0.094	SW	A.M. Fine and clear—light brisk wind. P.M. Cloudy.
♄ 23	29.835	66.7	29.800	67.7	55	63.3	64.4	52.3	67.4		SW	Cloudy.
○ ☉ 24	29.830	65.4	29.886	68.7	56	63.4	68.6	55.6	70.7	0.031	NE	Fair—cloudy.—Brisk wind A.M.
☽ 25	30.071	67.3	30.079	70.3	55	63.3	71.4	52.5	72.6		N	Fine—cloudy—light wind.
♂ 26	30.187	68.7	30.176	71.3	59	67.3	74.7	55.4	74.7		NNE	Fine—light clouds and wind.
☿ 27	30.245	69.6	30.209	73.3	61	70.6	79.2	59.7	79.7		ESE	Fine and cloudless.
♃ 28	30.176	71.6	30.117	74.7	62	69.6	77.7	64.7	79.3		NNW	Fine and clear—cloudy. Thunder at 7 P.M.
♀ 29	30.135	72.3	30.123	75.7	63	69.6	76.4	63.3	80.8		E	Clear—cloudy. Violent thunder-storm from 4 to 5 P.M.
♄ 30	30.192	72.7	30.157	75.3	64	67.6	77.4	61.3	78.7	0.436	N	A.M. Foggy. P.M. Fine and clear.
☉ 31	30.133	76.7	30.070	76.5	61	72.3	77.7	60.3	79.5		N	Fine and clear—light brisk wind.
	Mean 30.014	Mean 69.7	Mean 29.997	Mean 71.2	Mean 57.3	Mean 67.3	Mean 71.6	Mean 58.3	Mean 76.6	Sum 2.823		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. ………… { 9 A.M. 29.899 — 3 P.M. 29.878 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge …………………… =83 feet 2¼ in.

…………………………………… above the mean level of the Sea (presumed about) ………… =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House …………………… =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR AUGUST, 1831.

1831. August.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit.		Self-registering.				
						9 A.M.	3 P.M.	Lowest.	Highest.			
☽ 1	30.068	76.3	30.021	77.4	60	71.4	73.8	58.6	75.7		N	A.M. Cloudless. P.M. Fine. At night, rain.
♂ 2	29.876	71.3	29.878	75.3	66	66.3	70.0	63.6	73.3		N	Showery. Thunder at noon and in the evening.
☿ 3	29.903	71.3	29.893	74.2	64	66.7	70.6	63.7	72.3	0.367	ENE	Cloudy. Thunder at noon. Very heavy showers, P.M.
♃ 4	29.811	72.3	29.726	74.4	65	67.3	75.3	61.2	76.7	0.361	E	A.M. Overcast. P.M. Fine—cloudy.
♀ 5	29.682	71.7	29.689	75.4	69	68.7	74.4	65.3	75.3		E	Fair—cloudy. From 7 to 8 P.M. a violent thunder-storm.
♄ 6	29.760	73.6	29.740	75.7	64	69.4	74.3	60.4	76.3	0.375	SSW	Fine and clear—light clouds and wind.
● ☉ 7	29.687	74.7	29.684	75.0	64	70.4	70.7	62.3	75.7		ESE	A.M. Fair—cloudy. P.M. Lowering—heavy broken clouds—light rain.
☽ 8	29.859	70.3	29.869	74.6	64	64.0	76.8	62.0	76.8		NNW	Fine—lightly cloudy.
♂ 9	29.966	71.4	29.958	75.5	66	69.2	78.2	64.3	79.7		NW	Fine—cloudy. Rain at night.
☿ 10	30.057	71.3	30.099	73.5	58	65.6	72.4	61.3	74.3	0.203	N	Fine—light clouds.
♃ 11	30.192	72.7	30.152	73.8	59	68.4	73.6	59.7	74.6		N	A.M. Cloudless. P.M. Fine—light clouds.
♀ 12	30.163	70.3	30.123	73.4	63	65.3	72.8	60.3	75.8		E	A.M. Cloudless. P.M. Fine—light clouds and wind.
♄ 13	30.080	72.6	30.014	73.7	62	68.6	72.5	60.7	75.6		NNW	A.M. Cloudless. P.M. Lightly overcast—shower at 2.
☉ 14	30.040	71.7	30.005	73.4	60	65.3	71.7	56.7	72.6	0.042	N	Fine and clear—light clouds.
☽ 15	30.107	70.7	30.108	72.7	61	65.5	71.6	55.7	73.2		N	Fine and cloudless.
♂ 16	30.140	68.3	30.104	70.6	62	62.7	69.4	57.7	70.2		E	A.M. Strong fog. P.M. Overcast—thunder.
☿ 17	30.080	69.7	29.998	72.5	64	65.7	72.5	60.8	74.3	0.017	W	Fine—cloudy. Violent thunder-storm, with rain at 5 P.M.
♃ 18	29.970	68.3	29.944	69.4	49	60.3	67.8	66.3	68.3	0.264	N	Clear and cloudless.
♀ 19	29.678	66.3	29.719	69.6	64	66.9	66.4	56.4	69.7	0.014	WSW	Fine and clear.
♄ 20	29·666	66.6	29·812	68.6	59	61.3	67.5	56.7	67.7	0.011	N	Fine—brisk unsteady wind and light clouds. Rain at night.
☉ 21	30.096	65.7	30.164	70.3	61	62.6	67.5	59.4	68.4	0.014	N	Fine—cloudy—brisk unsteady wind.
☽ 22	30.310	64.7	30.277	69.2	59	60.7	68.0	55.3	68.7		NNW	Fair—cloudy.
○ ♂ 23	30.202	67.7	30.180	71.6	58	66.4	73.8	60.8	75.6		N	Fair—light clouds.
☿ 24	29.977	69.7	29.869	72.2	62	67.6	70.4	58.7	72.6		SSW	Lightly overcast.
♃ 25	29.730	67.9	29.794	70.5	52	62.7	68.8	59.7	69.5	0.033	W	Fine—light clouds. Clear A.M.
♀ 26	29.983	70.3	29.946	71.6	54	66.7	69.8	54.4	71.5		SSW	Lightly overcast.
♄ 27	29.975	69.7	29.961	72.4	65	68.7	72.4	61.7	75.3		WSW	Lightly overcast.
☉ 28	30.129	70.3	30.160	71.7	61	63.6	69.5	55.7	70.5		WSW	Fine—nearly cloudless.
☽ 29	30.239	69.4	30.202	71.0	59	63.7	72.2	55.3	72.7		WSW	Fine. A.M. Cloudless.
♂ 30	30.147	69.3	30.101	72.4	63	66.5	73.0	59.6	74.3		SSW	Lowering.
☿ 31	29.952	69.7	29.948	72.2	66	68.4	68.0	63.7	70.5		WSW	Fair—cloudy.
	Mean 29.985	Mean 70.2	Mean 29.972	Mean 72.7	Mean 61.4	Mean 66.0	Mean 71.5	Mean 59.9	Mean 73.2	Sum 1.701		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr. .... { 9 A.M. 29.869 — 3 P.M. 29.848 }

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................ =83 feet 2½ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR SEPTEMBER, 1831.

1831. Septemb.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
						Fahrenheit.		Self-registering.				
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		9 A.M.	3 P.M.	Lowest.	Highest.			
♃ 1	29.966	66.0	29.888	63.7	58	58.8	53.5	55.6	59.4	0.041	SW	Rain.
♀ 2	29.843	62.0	29.883	64.6	55	55.0	58.7	48.2	60.4	0.922	SW var.	Fine—light clouds and wind.
♄ 3	29.918	60.6	29.883	63.0	52	55.4	62.5	47.0	63.2		NNE	Fine and clear.
☉ 4	29.847	61.6	29.839	64.3	55	58.3	65.0	47.2	68.2		SSW	A.M. Fine. P.M. Light rain.
☽ 5	29.911	64.0	29.924	68.9	57	64.2	70.2	58.5	72.2	0.030	S	A.M. Light rain. P.M. Fine—light clouds.
● ♂ 6	29.926	66.0	29.871	68.0	60	62.8	65.9	51.6	67.4	0.038	SW	A.M. Overcast. P.M. Fine—light clouds.
☿ 7	29.879	65.0	29.788	66.7	55	60.3	62.2	51.8	64.8	0.061	S	A.M. Nearly cloudless. P.M. Light rain.
♃ 8	29.713	62.6	29.649	65.8	52	58.7	62.0	48.2	65.9	0.230	SW	A.M. Fine. P.M. Light rain.
♀ 9	29.584	60.3	29.643	62.8	54	55.2	55.5	50.0	58.2	0.058	SW	Overcast—light rain.
♄ 10	29.845	60.3	29.881	63.6	55	57.9	60.3	53.0	63.6	0.144	NW var.	A.M. Fair. P.M. Rain.
☉ 11	29.984	58.8	30.051	62.5	52	55.6	61.2	49.4	61.7	0.108	S	Cloudy—rain.
☽ 12	30.202	60.8	30.210	63.4	55	58.8	63.7	55.9	64.6		S	A.M. Fair. P.M. Cloudy.
♂ 13	30.214	61.0	30.176	64.5	54	58.4	64.8	54.6	66.0		ENE	Fine—lightly overcast.
☿ 14	30.158	61.7	30.132	64.2	53	59.5	63.6	55.3	64.5		E	A.M. Fine. P.M. Cloudy.
♃ 15	30.172	60.2	30.138	64.0	52	55.5	63.7	51.0	66.2		E	Fine and clear.
♀ 16	30.249	60.4	30.273	64.2	52	55.2	61.4	54.0	62.6		NE	Overcast. P.M. Light rain.
♄ 17	30.239	59.9	30.208	63.6	52	57.5	60.3	53.2	64.3		WSW	A.M. Fine. P.M. Overcast.
☉ 18	30.120	61.0	30.049	64.6	53	58.4	64.8	54.0	66.8		SSW	A.M. Fair. P.M. Overcast.
☽ 19	29.917	61.4	29.860	65.4	54	58.6	64.8	52.2	66.8		ESE	A.M. Fine. P.M. Cloudy. At night rain.
♂ 20	29.859	59.4	29.851	63.4	51	54.2	62.0	46.9	63.5	0.041	SW	Fine and clear.
○ ☿ 21	29.814	61.2	29.825	64.5	56	58.8	64.0	54.0	65.2	0.144	SW	A.M. Light rain. P.M. Fine and clear.
♃ 22	29.934	60.3	29.949	63.4	50	54.8	62.6	51.0	63.6		WSW	Fine—light clouds.
♀ 23	30.130	59.0	30.118	63.2	52	54.2	64.8	46.5	65.2		W	Fine—light clouds.
♄ 24	30.241	62.4	30.173	65.3	57	60.6	66.6	54.5	68.2		S	Fine and clear—light clouds.
☉ 25	30.051	63.2	29.976	65.4	57	61.2	66.7	54.7	67.6		S var.	Fine—nearly cloudless P.M.
☽ 26	30.039	63.0	29.986	64.5	56	58.2	62.2	58.0	63.8	0.033	S	Overcast—light rain.
♂ 27	29.917	62.2	29.836	65.5	54	57.8	66.4	55.3	66.2		E	A.M. Light rain. P.M. Nearly cloudless.
☿ 28	29.775	63.8	29.724	66.6	57	61.7	67.5	57.4	68.0	0.325	E	Fine—light clouds. At night, thunder-storm.
♃ 29	29.629	65.0	29.609	68.3	60	61.2	67.8	59.6	69.2	0.631	E	Fair—lightly overcast.
♀ 30	29.540	65.0	29.433	68.0	61	62.6	68.3	58.5	69.5		SE	A.M. Fine. P.M. Cloudy. At night, thunder-storm.
	Mean 29.954	Mean 61.9	Mean 29.928	Mean 64.9	Mean 54.7	Mean 58.3	Mean 63.4	Mean 52.9	Mean 65.2	Sum 2.806		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr.	9 A.M.	3 P.M.
	29.863	29.828

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................... =83 feet 2½ in.

.... above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR OCTOBER, 1831.

1831. Octob.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
						Fahrenheit.		Self-registering.				
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		9 A.M.	3 P.M.	Lowest.	Highest.			
♄ 1	29.311	66.2	29.340	68.7	62	63.0	66.6	61.0	68.2	0.261	S	Fine—light clouds. Heavy rain at night.
☉ 2	29.365	65.2	29.455	67.6	60	61.6	65.4	56.7	66.4	0.188	SE var.	Fine—light clouds.
☽ 3	29.720	64.0	29.781	66.8	61	60.0	65.0	54.2	66.4		S	Fine and clear—light wind. Rain at night.
♂ 4	29.842	64.4	29.917	66.0	58	58.8	63.6	56.6	64.7	0.158	WSW	Fine and cloudless. Heavy rain at night.
● ☿ 5	29.931	63.5	29.974	65.5	58	58.4	62.2	55.3	63.4	0.119	WSW	Fine and cloudless.
♃ 6	29.951	62.7	29.934	65.4	60	61.0	64.8	54.4	66.3		S var.	Cloudy—light unsteady wind. Heavy rain at noon.
♀ 7	29.754	64.3	29.691	66.6	61	63.8	66.6	59.3	67.8	0.033	ESE	Fine and cloudless.
♄ 8	....	....	29.643	63.9	...	......	58.6	57.0	61.2		SE	Overcast—light rain. Thunder-storm at 1 P.M.
☉ 9	29.683	62.3	29.724	64.3	58	56.5	61.0	53.0	61.6	0.463	WSW	Fair—lightly cloudy.
☽ 10	29.667	61.2	29.647	63.5	57	58.0	60.7	53.0	63.0	0.003	SW	Fine—cloudy—light wind.
♂ 11	29.732	61.5	29.758	64.0	58	59.3	60.6	56.6	62.2		S	A.M. Fine—light clouds. P.M. Light rain.
☿ 12	29.750	61.2	29.720	62.8	59	58.7	61.6	56.7	62.2		NW	Overcast—light rain.
♃ 13	29.697	61.4	29.633	64.6	58	60.0	62.5	54.6	64.3	0.725	SSE	Overcast—light wind. High wind through the night,
♀ 14	29.615	63.7	29.677	65.5	61	61.7	62.7	60.7	64.3		SSE var.	Lowering—unsteady wind.
♄ 15	29.822	62.7	29.857	65.4	58	58.6	61.8	56.3	63.3		SSW	Fair—lightly overcast.
☉ 16	30.185	59.4	30.251	62.3	52	52.7	60.3	48.2	60.3		WSW	Fine and clear—light clouds.
☽ 17	30.383	58.6	30.364	61.7	55	55.4	60.8	47.6	61.5		W	Fine and clear.
♂ 18	30.432	59.8	30.404	61.8	58	58.4	60.7	55.3	62.3		W	Strong haze.
☿ 19	30.266	60.7	30.174	62.8	57	57.8	62.8	55.3	63.6		E	Fine and cloudless.
♃ 20	30.017	60.7	29.978	63.3	58	58.4	60.8	53.3	62.3		S	Cloudy.
○ ♀ 21	29.951	60.9	29.962	62.7	57	57.5	59.0	55.3	59.5		SW	Cloudy. Evening, clear.
♄ 22	30.066	55.6	29.981	58.5	50	53.7	56.4	46.7	58·7		SSE	Overcast. Rain P.M.
☉ 23	29.960	59.3	29.816	60.7	57	58.8	61.0	53.3	61.0		S	Cloudy. P.M. Rain. Evening, clear.
☽ 24	30.099	57.3	30.058	60.2	48	51.7	58.0	47.6	58.6		WSW	Fine. A.M. Cloudless.
♂ 25	29.891	58.0	29.764	59.3	53	56.4	55.8	51.4	58.4		SSE	Cloudy. P.M. Light rain.
☿ 26	29.536	58.5	29.572	60.2	57	57.4	57.8	53.3	59.7		SSE	Cloudy. Light shower A.M.
♃ 27	29.742	57.3	29.691	58.0	52	52.8	54.7	52.3	55.0	0.119	SW	Rain.
♀ 28	30.077	55.8	30.204	58.0	52	52.7	56.5	48.4	57.3	0.036	SW	Fine and cloudless.
♃ 29	30.314	55.4	30.267	58.2	51	51.6	56.3	48.3	57.2		W	A.M. Cloudless—fog and deposition. P.M. Cloudy. Evening clear.
☉ 30	30.302	52.6	30.245	55.4	44	44.2	52.7	40.3	52.7		WSW	A.M. Cloudless—haze and deposition. P.M. Lightly cloudy.
☽ 31	30.208	53.9	30.176	56.6	51	51.7	56.2	43.5	57.4		WSW	A.M. Fine. P.M. Lightly overcast.
	Mean 29.909	Mean 60.3	Mean 29.892	Mean 62.6	Mean 56.0	Mean 57.0	Mean 60.4	Mean 53.1	Mean 61.6	Sum 2.105		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr.	9 A.M.	3 P.M.
	29.822	29.798

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2½ in.

.... above the mean level of the Sea (presumed about)................ =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR NOVEMBER, 1831.

1831. Novemb.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
						Fahrenheit.		Self-registering.				
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		9 A.M.	3 P.M.	Lowest.	Highest.			
♂ 1	30.035	57.4	29.950	55.8	51	53.8	55.2	......	55.3		S	Cloudy.
☿ 2	29.777	55.6	29.730	56.6	53	53.5	52.5	52.3	53.4		WSW	Fine—lightly cloudy.
♃ 3	29.494	52.6	29.449	52.9	44	46.7	45.0	44.3	48.7		SW	Fine—lightly cloudy.
● ♀ 4	29.771	47.6	29.849	50.3	42	42.5	46.8	35.6	47.2		W	A.M Cloudless. P.M. Lightly overcast.
♄ 5	29.520	47.6	29.410	50.7	47	47.4	50.5	38.3	51.3		S	Overcast. Light rain P.M.
☉ 6	29.607	47.3	29.385	49.7	44	44.3	50.2	39.7	56.3		WSW	Foggy. Rain P.M.
☽ 7	29.477	50.0	29.527	51.8	41	46.9	49.4	43.4	50.3	0.025	SW	Fine and cloudless.
♂ 8	29.666	49.3	29.784	51.9	45	46.8	51.2	43.4	51.3		WSW	Fine and cloudless.
☿ 9	30.155	46.4	30.245	48.2	39	39.4	46.2	37.4	46.7		W	A.M. Strong fog. P.M. Fine.
♃ 10	30.421	43.4	30.333	45.2	35	35.3	41.8	32.4	48.7		WNW	Foggy.
♀ 11	30.245	47.4	30.255	49.8	49	49.4	53.6	34.4	53.6	0.033	WSW	Foggy. Evening, light rain.
♄ 12	30.396	51.0	30.348	54.0	51	52.7	54.3	48.7	55.3		NNW	Foggy.
☉ 13	29.994	51.8	29.968	52.7	49	49.7	49.4	48.4	49.7		W	A.M. Foggy—deposition. P.M. Fine—light clouds.
☽ 14	30.053	45.3	29.762	47.8	32	36.4	43.6	32.8	44.4		WSW	A.M. Cloudless—light wind. P.M. Hazy. At night, rain.
♂ 15	29.478	44.2	29.372	45.5	35	37.4	40.4	34.7	40.6	0.033	W	A.M. Cloudless. P.M. Lightly overcast.
☿ 16	29.312	41.4	29.372	43.0	33	36.3	40.4	32.7	40.4		WNW	A.M. Cloudless. P.M. Fair—light haze. Evening, light rain.
♃ 17	29.585	39.7	29.546	40.5	33	33.8	36.4	31.7	36.4	0.239	W	Foggy. Light snow early A.M.
♀ 18	29.691	38.5	29.765	40.8	29	31.3	37.6	28.7	38.3		WNW	Overcast—foggy.
○ ♄ 19	29.436	40.7	29.493	43.2	40	40.4	43.5	30.3	43.5	0.089	NNW	Fair—light clouds and wind.
☉ 20	29.838	40.7	29.882	42.3	35	38.6	40.3	33.7	52.3		W	Overcast—light fog. Rain at night.
☽ 21	29.679	44.3	29.776	46.9	52	52.4	54.7	37.4	55.3	0.283	WNW	Overcast—foggy—light wind. Evening, deposition.
♂ 22	29.850	48.8	29.893	50.7	54	54.6	57.2	51.8	57.2	0.158	WSW	Foggy. A.M. Deposition. Evening, light rain.
☿ 23	29.966	52.7	29.962	54.3	53	54.6	56.8	53.0	56.7		WSW	Overcast.
♃ 24	30.032	53.4	30.004	55.4	50	50.6	56.8	47.7	56.8		SW	Fine—light clouds.
♀ 25	29.960	54.3	29.893	55.5	51	51.2	54.4	47.0	54.4		SSW	Cloudy. Evening, light rain.
♄ 26	29.944	54.6	30.051	53.7	51	51.5	52.5	50.3	52.5		W	Overcast.
☉ 27	30.321	46.7	30.367	47.7	32	37.2	41.4	35.6	41.6		NE	Overcast.
☽ 28	30.497	43.9	30.480	44.6	29	35.8	41.2	33.7	41.2		NNW	Foggy—light wind.
♂ 29	30.547	42.7	30.526	42.8	36	36.7	43.2	36.3	47.8		E	Foggy.
☿ 30	30.314	41.7	30.190	43.3	36	36.7	42.8	34.3	46.4		W	Light fog. Rain P.M.
	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Mean	Sum		
	29.900	47.4	29.886	48.9	42.4	49.8	47.6	39.3	49.1	0.860		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32°Fahr.	9 A.M.	3 P.M.
	29.851	29.832

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge .... =83 feet 2½ in.
.... above the mean level of the Sea (presumed about) .... =95 feet.
The external Thermometer is 2 feet higher than the Barometer Cistern.
Height of the Receiver of the Rain Gauge above the Court of Somerset House .... =79 feet 0 in.
The hours of observation are of Mean Time, the day beginning at Midnight.
The Thermometers are graduated by Fahrenheit's Scale.
The Barometer is divided into inches and decimals.

METEOROLOGICAL JOURNAL FOR DECEMBER, 1831.

1831. Decemb.	9 o'clock, A.M.		3 o'clock, P.M.		Dew Point at 9 A.M. in degrees of Fahr.	External Thermometer.				Rain, in inches. Read off at 9 A.M.	Direction of the Wind at 9 A.M.	Remarks.
	Barom.	Attach. Therm.	Barom.	Attach. Therm.		Fahrenheit.		Self-registering.				
						9 A.M.	3 P.M.	Lowest.	Highest.			
♃ 1	30.170	46.8	30.125	47.3	44	44.3	46.8	37.3	47.3	0.078	NW	Overcast—light wind.
♀ 2	30.027	46.7	30.027	48.4	44	45.0	47.5	44.3	47.5		NNW	Overcast and foggy.
♄ 3	30.148	47.7	30.128	47.9	43	44.7	45.8	44.6	45.8		NW	Foggy—light wind.
● ☉ 4	30.028	47.7	29.658	49.3	45	45.7	49.3	44.6	49.7		WSW	Foggy—light deposition.
☽ 5	29.837	48.8	29.720	49.9	......	44.6	48.2	44.3	48.3		WSW	Fair—light fog.
♂ 6	29.432	48.8	29.421	50.3	47	47.5	48.8	43.0	51.3		SW	Fair—lightly overcast—light wind.
☿ 7	28.924	50.7	28.986	52.4	50	51.3	52.0	47.6	53.3		SSE	Overcast. High wind with rain A.M.
♃ 8	29.139	51.3	29.124	53.4	51	51.3	54.3	47.8	54.0	0.089	SSW	Light rain and fog.
♀ 9	29.190	55.3	29.206	56.3	53	53.4	53.5	51.6	54.6		SW	A.M. Clear. P.M. Rain.
♄ 10	29.418	53.8	29.569	55.8	......	50.7	51.7	46.7	52.4		S	Fair—lightly overcast.
☉ 11	29.342	53.9	29.309	55.5	50	52.6	54.2	48.8	55.6	0.305	SSE var.	Light wind. A.M. Fair. P.M. Overcast
☽ 12	29.307	53.8	29.128	54.6	51	51.2	54.0	49.0	54.2		SE var.	Overcast—light wind. Light rain A.M.
♂ 13	29.395	53.2	29.497	54.4	48	50.0	51.0	48.9	51.2		SSW	A.M. Fine. P.M. Light rain.
☿ 14	29.573	51.5	29.582	52.2	47	46.8	47.7	44.5	48.6		SW	Fair—light haze and wind.
♃ 15	29.792	47.7	29.742	49.5	38	39.3	45.8	38.4	47.3		WSW	A.M. Cloudless—fog. P.M. Lightly overcast.
♀ 16	29.851	47.7	29.606	50.4	37	38.5	47.6	37.7	50.7		SW	A.M. Cloudless—fog and deposition. High wind with rain at night.
♄ 17	29.744	48.4	29.794	49.6	37	40.4	44.0	38.7	47.7		WSW	Cloudless—hazy.
☉ 18	29.373	49.5	29.412	50.3	44	44.8	45.5	40.7	46.4	0.033	SSW	A.M. Fog and deposition. P.M. Fine.
○ ☽ 19	29.567	46.7	29.643	47.6	39	39.9	42.5	39.7	42.6		WSW	Cloudless—hazy.
♂ 20	29.775	45.6	29.707	47.0	37	40.3	43.7	39.7	44.7		SW	Light fog and wind.
☿ 21	29.657	46.0	29.742	46.7	42	42.2	43.2	40.5	43.7		W	Fog and wind.
♃ 22	29.836	44.7	29.649	46.8	40	40.0	46.3	35.8	46.4		SSW	Lightly overcast. Deposition A.M.
♀ 23	29.815	44.7	29.980	46.2	36	38.7	41.3	36.5	41.3		WSW	Foggy—light wind.
♄ 24	30.233	42.2	30.262	42.2	33	35.3	35.0	32.9	38.0		......	Strong fog throughout the day.
☉ 25	30.331	39.7	30.286	39.7	32	32.5	33.6	30.3	34.4		WNW	A.M. Strong haze. P.M. Lightly cloudy.
☽ 26	30.274	38.7	30.311	40.2	34	34.4	36.8	29.8	37.3		WNW	Strong haze, A.M. and evening. Fine, P.M.
♂ 27	30.427	39.0	30.418	40.0	33	35.7	38.3	33.3	38.3		......	Foggy—light wind.
☿ 28	30.428	40.4	30.392	42.7	39	39.4	41.0	35.3	41.3		NNE	Fair—light fog.
♃ 29	30.313	41.4	30.229	43.5	38	39.5	40.0	37.3	40.7		NNE	A.M. Overcast. P.M. Fair.
♀ 30	30.226	41.2	30.186	41.0	33	36.2	35.9	35.3	36.3		NNE	Fine—light clouds and haze.
♄ 31	30.238	38.4	30.227	39.6	33	34.6	35.8	32.3	35.7		N	Fine and cloudless—light haze.
	Mean 29.800	Mean 46.9	Mean 29.776	Mean 48.1	Mean 41.3	Mean 42.9	Mean 45.2	Mean 40.6	Mean 46.0	Sum 0.505		

Monthly Mean of the Barometer, corrected for Capillarity and reduced to 32° Fahr.	9 A.M.	3 P.M.
	29.751	29.724

OBSERVANDA.

Height of the Cistern of the Barometer above a Fixed Mark on Waterloo Bridge........................... =83 feet 2⅓ in.

....above the mean level of the Sea (presumed about) .... =95 feet.

The External Thermometer is 2 feet higher than the Barometer Cistern.

Height of the Receiver of the Rain Gauge above the Court of Somerset House.................. .... =79 feet 0 in.

The hours of observation are of Mean Time, the day beginning at Midnight.

The Thermometers are graduated by Fahrenheit's Scale.

The Barometer is divided into inches and decimals.